A Guided Tour from Linear Algebra to the Foundations of Gabor Analysis

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It is the aim of this chapter to provide motivation and technical background on some of the key topics in Gabor analysis: the Gabor frame operator, the notion of (dual) Gabor frames, the Janssen representation of the Gabor frame operator, and the spreading function as a tool to describe operators.

Starting from the context of finite dimensional signal spaces, as it can be constructively realized on a computer and completely described in terms of concepts from linear algebra, we describe first the algebraic side of this problem. The second part of the material emphasizes the necessary definitions and problems (e.g., the form of convergence of various infinite series occurring during the discussion) and provides a description of tools that have become relevant within modern time-frequency analysis in order to overcome the technical problems arising in the context of Gabor analysis involving a continuous domain (such as the Euclidean space). As it is shown, the so-called Banach Gelfand Triple built upon the Segal Algebra $S_0(\mathbb{R}^d)$ appears as the appropriate and universally useful tool for a clear treatment of questions of time-frequency analysis, even if the interest is mostly in operators over L^2. On the other hand a family of function (or distribution) spaces defined by means of time-frequency tools (the so-called modulation spaces) appear to provide a suitable frame-work for a more refined discussion of mathematical problems arising in this field.

1. Introduction

The goal of this report is to provide an introduction to Gabor analysis, from basic principles of linear algebra up to the foundations of advanced time-frequency concepts for Gabor Analysis. In order to make the text accessible both to engineers interested in the discrete side of Gabor analysis as well as to mathematicians with a general background in functional analysis there will be essentially two ways to read the article. One path leads directly from linear algebra to "finite Gabor Analysis" (more precisely, Gabor analysis for vectors of finite length m, which however are interpreted as periodic sequences, resp. functions over the group of unit roots of order m), and exploits the purely algebraic structure of Gabor systems. This rich structure (e.g., the commutation relation for the frame matrix) implies the possibility to efficiently compute the dual Gabor atom, respectively the bi-orthogonal system, for a linearly independent Gabor family. Note that it is not far from this problem to the discussion of applications of Gabor analysis in the framework of wireless communication (OFDM [44] etc.). On the other hand the numerical realization of the corresponding steps already leads to discussions about the size of certain constants (i.e., the condition number of the frame operator) which may be large (but still finite) in the finite dimensional setting, but could be infinite in similar situations for functions on the real line. Obviously, there the natural domain for Gabor Analysis are the square integrable functions, i.e., the Hilbert space $L^2(\mathbb{R})$ respectively in several dimensions $L^2(\mathbb{R}^d)$.

In this context we are facing three additional aspects. The first one is the infinite-dimensionality of the underlying space, and the necessary distinction between bounded and unbounded linear operators. As a consequence there are injective mappings (such as the coefficient mapping with respect to some family of elements in a Hilbert space) which do *not* have a closed range, for example. Indeed, the Gabor system as suggested by D. Gabor in 1946 is such an example (of a total system of vectors, which is *not a frame*), see [20]. These two possible shortcomings of a "generating system" result in the need to require the existence of two constants (essentially equivalent to the boundedness of the frame operator and its inverse). These requirements lead to the concepts of a *frame* and *Riesz bases* respectively, which however are by now quite well understood and can be handled using standard functional analytic concepts.

The second difficulty stems from the fact that - unlike the continuously defined short time Fourier transform (STFT) - a sampled STFT (the starting point of Gabor Analysis) for general L^2-windows comes in conflict with the (boundedness) requirements coming in naturally on functional analytic reasons as discussed above. Moreover, one expects in a continuous domain that dual Gabor windows depend continuously on the Gabor atom and the lattice constants in use, and also that the sum describing the frame operator in the Janssen representation converges absolutely. All this can be granted by assuming that atoms are exclusively taken from the Segal Algebra $S_0(\mathbb{R}^d)$, also called Feichtinger's algebra in the literature. It may be not surprising that this space in turn can be defined by means of the STFT, and that it is characterized in the context of Gabor analysis as the collection of those continuous and integrable functions which have - with respect to almost any "nice" Gabor frame - an absolutely convergent Gabor expansion. We will devote some paragraph to the discussions of this space and its properties, but even more so to point out how useful it is in the context of Gabor analysis (and time-frequency analysis in general). It shares many essential properties with the more widely known Schwartz space of "rapidly decreasing functions". Among others it is Fourier invariant, and hence its dual space is a natural domain for an extended Fourier transform. On the other hand it is a simple Banach space with respect to a very natural norm and provides an answer to the problems mentioned above (and many more). The "simple distributions" (one might call the elements of $S_0(\mathbb{R}^d)$ by this name) are however good enough to prove a kernel theorem and to establish a so-called Banach Gelfand Triple (BGT), consisting of the Banach space of test functions $S_0(\mathbb{R}^d)$, the (usual) Hilbert space $L^2(\mathbb{R}^d)$ and the (simple) distribution space $S_0'(\mathbb{R}^d)$. It is quite intuitive to think of these three layers "melted together" in the finite dimensional case, but essentially different in the continuous domain. For example, one can describe the Fourier transform on $\mathbb{R}^d$ as a (unitary) Gelfand triple isomorphism, with the nice fact that the ordinary integral representation of the Fourier transform makes sense (and also the inversion theorem in the pointwise sense) for functions from $S_0(\mathbb{R}^d)$. On the other hand, its behavior at the level of the Hilbert space allows to express the fact that it is unitary, hence preserving angles and L^2-norms. Finally one can claim (at the distributional level) that the Fourier transform maps "pure frequencies" onto the corresponding Dirac-measures (which take of course the role of unit vectors), and that the Fourier transform is uniquely determined (as Gelfand triple isomorphism) by this fact. There are more such Gelfand triple isomorphisms, but we will exploit only the "spreading mapping" a bit more closely, in order to describe the fact that every linear mapping (from S_0 to S_0') is a superposition of TF-shift operators (in a certain sense), while any Gabor frame operator is a (discrete) sum of TF-shifts (the abstract version of Janssen's theorem). In this context we will heavily make use of the so called fundamental relation for Gabor Analysis (FIGA).

The third new point arising in the context of the real line (as opposed to the finite dimensional situation, where any two norms are equivalent) is the need to make a distinction between "good" and "nasty" functions, e.g. using notations related to summability (L^p-spaces in the classical setting of Fourier analysis), smoothness or decay at infinity. One expects that a good frame consisting of "nice atoms" has the property that smooth functions with strong decay at infinity require only a small number of non-zero terms for a good approximation. This is perfectly realized for the classical Besov – Triebel – Lizorkin spaces with respect to "good" wavelet systems and the complete characterization of those spaces via wavelet coefficients is an important aspect of modern wavelet theory. In turn, the "correct" function spaces in the time-frequency context are modulation spaces, introduced in the early eighties. Since they are exactly the spaces of distributions with Gabor coefficients in ℓ^p, the modulation spaces $M^0_{p,p}$ play an important role in time-frequency analysis. As with the case of the classical theory of L^p-spaces the three special cases $p = 1, 2, \infty$ are the most interesting ones, and can be described with less technical effort compared to the more general, by now "classical modulation spaces" $M^s_{p,q}$, which have been modelled in analogy to standard Besov spaces. Since $M^0_{1,1}$ is just the Segal algebra S_0 and its dual $M^0_{\infty,\infty}$ coincides with S_0', we will restrict our attention to these spaces, which make up a *Banach Gelfand Triple* together with $M^0_{2,2} = L^2$. Let us mention however here that this restriction was chosen in order to limit the level of technicality of our presentation, although many of the results involving these spaces carry over to the more general modulation spaces (derived from general solid and translation invariant spaces over phase space) as described by the general coorbit theory developed by Feichtinger and Gröchenig in the late eighties [12].

2. Basics in linear algebra

Assume that a signal f is a (column) vector of m complex numbers. Given a family of (column) vectors $g_1, \ldots, g_n$ in $\mathbb{C}^m$ we are looking for a linear combination of these vectors that reproduces f or comes close to f. In other words, we are looking for a coefficient vector c in $\mathbb{C}^n$ such that

$$f \ \sim \ c(1)g_1 + \cdots + c(n)g_n \,. \tag{1}$$

We can transfer this problem to matrix notation. Let A be the matrix of size $m \times n$ whose j-th column is g_j. Then equation (1) boils down to the matrix vector product

$$f \ \sim \ Ac \,.$$

In the context of the linear mapping $c \mapsto Ac$ induced by A from $\mathbb{C}^n$ (the coefficient space) to $\mathbb{C}^m$ (the signal space), there exist coefficient vectors exactly reproducing f if and only if f is in the range of A, write $R(A)$, which obviously coincides with the linear span of $g_1, \ldots, g_n$ respectively the column-space of A.

So far we have defined the problem and built a simple mathematical model. In order to obtain a better understanding of our problem we need a few basic facts from linear algebra.

A family of vectors $g_1, \ldots, g_n$ is said to be *linearly independent* if

$$a(1)g_1 + \cdots + a(n)g_n = 0$$

implies $a(1) = ... = a(n) = 0$. Otherwise the family is *linearly dependent*. Obviously linear independence is another way of expressing the injectivity of the linear mapping induced by A. The *rank* of a matrix A, write $r(A)$, is the maximal number of linearly independent rows. Important properties are:

(1) $r(A) = r(A')$ where A' denotes the *transpose conjugate* of A.
(2) $r(A)$ coincides with the dimension of $R(A)$.
(3) A of size $m \times n$ is *surjective*, i.e., $R(A) = \mathbb{C}^m$, if and only if $r(A) = m$.

We see that imposing $r(A) = m$ requires $n \geq m$ which is always the case of frames discussed below. In this case, for *every* $f \in \mathbb{C}^m$, there exists at least one coefficient vector c such that $Ac = f$. We are interested in how to find such a coefficient vector.

An easy exercise shows that the nullspaces of A and AA' coincide and therefore A has maximal rank $r(A) = m$ if and only if AA' is invertible. For $r(A) = m$ it follows that $A'(AA')^{-1}$ is well-defined and we can see that

$$c \ = \ A'(AA')^{-1}f$$

provides a suitable coefficient vector for reproducing f.

The matrix $A'(AA')^{-1}$ is known as the *pseudo-inverse* of A (whose size is $n \times m$) [23]. Every matrix has a pseudo-inverse which can be computed by the singular value decomposition. In the singular value decomposition the matrix A is decomposed as

$$A = U\Sigma V' ,$$

where the (first $r(A)$) columns of the matrix U are an orthonormal system for the range of A, Σ is a diagonal matrix with the singular values of A (i.e., the eigenvalues of AA') in its diagonal, and V is a well-chosen orthonormal system of $\mathbb{C}^n$. An excellent introduction for the singular value decomposition can be found in [4]. The standard approach for computing the singular value decomposition of a $m \times n$ matrix A ($m \leq n$) is as follows:

(1) Compute the eigenvalues of $AA' = V\Lambda V'$.
(2) Let Σ be the $m \times n$ nonnegative square root of Λ.
(3) Solve the system $U\Sigma = AV$ for unitary U (e.g., via QR-factorization).

The above case with $r(A) = m$ and $m \leq n$ is just a special case for computing a pseudo-inverse.

The question about the uniqueness of the coefficient vector c can be answered as follows. Whenever $n > m$, the family $g_1, \ldots, g_n$ is linearly dependent and any g can be written as a linear combination of the others. Therefore we could, for instance, change one single coefficient and adapt the others such that we recover f. Hence, c is not unique. The coefficients obtained via the pseudo-inverse are of minimal ℓ^2-norm (minimal length). In other words, the pseudo-inverse, applied to the right hand side of a linear problem of the form $Ac = f$ provides the minimal least square solution to this problem.

On the other hand, if $m = n$, then A (assumed to be surjective) is also injective and the difference between c and any other coefficient vector reproducing f must be in the kernel of A which contains only the zero vector. Hence c is unique. Equivalently we could argue that if $m = n$ then the family $g_1, \ldots, g_n$ is linearly independent and, thus, it constitutes a basis for $R(A)$ allowing only unique representations. In the latter case, the pseudo-inverse turns into the standard inverse of A.

Let us come back to the matrix product $A'(AA')^{-1}$. If we consider the complex conjugate of the rows of the pseudo-inverse $A'(AA')^{-1}$ and write them as column vectors, we obtain another family of n vectors in $\mathbb{C}^m$, say $\tilde{g}_1, \ldots, \tilde{g}_n$. As a consequence, we have for all $f \in \mathbb{C}^m$

$$f \;=\; AA'(AA')^{-1}f \;=\; \sum_{j=1}^{n} \langle f, \tilde{g}_j \rangle g_j \,, \tag{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{C}^n$. By interchanging the roles of $g_1, \ldots, g_n$ and $\tilde{g}_1, \ldots, \tilde{g}_n$, a simple argument yields

$$f \;=\; (AA')^{-1}AA'f \;=\; \sum_{j=1}^{n} \langle f, g_j \rangle \tilde{g}_j \,. \tag{3}$$

This shows that the families $\{g_j\}$ and $\{\tilde{g}_j\}$ are dual to each other in the sense that one system provides a left inverse of the other. In the case $m = n$ this is just the dual basis, which in the case of an orthonormal basis coincides with the original system, which amounts to A being unitary.

Besides decomposing a signal, the question of recovering a signal from a coefficient vector is of equal interest in application. That is, given coefficients of the form $\langle f, g_j \rangle$ of a signal f with respect to some basic family $\{g_j\}$, we want to recover the signal f. Fortunately, this can be treated as the dual problem of finding the right coefficients for the corresponding representation. Indeed, in the case where the analyzing family $\{g_j\}$ constitutes a frame, we recover the signal by using a dual frame $\{\tilde{g}_j\}$ as shown in (3).

For practical applications, the numerical properties of the analysis and synthesis mapping are of great importance. These are reflected in the singular values of the matrix A. The singular values coincide with the square root of the eigenvalues of the positive matrix AA'. Specifically, we are interested in the quotient between the maximal $a_{\max}$ and the minimal singular value $a_{\min}$. This quotient is the *condition number* of A and indicates the computational goodness of the algebraic system $Ac = f$.

Let us consider, for example, the family $e_1, e_1 + 10^{-k}e_2$, (e_1, e_2 denote the standard unit vectors in $\mathbb{C}^2$) which is obviously linearly independent in $\mathbb{C}^2$. For constructing e_2, the length of the coefficients is more than 10^k times larger compared to the length of the coefficients for the standard basis. This family is rather badly conditioned.

The minimal and maximal singular value of A are the optimal constants in the following inequality:

$$a_{\min}\|f\|^2 \;\le\; \langle AA'f, f\rangle \;=\; \sum_{j=1}^{n} |\langle f, g_j\rangle|^2 \;\le\; a_{\max}\|f\|^2 \qquad (4)$$

for all $f \in \mathbb{C}^m$. Inequality (4) is the so-called *frame inequality* (stated in the finite setting). Any family $g_1, \ldots, g_n$ satisfying (4) is called a *frame* for $\mathbb{C}^m$ and the lower and upper bounds are the corresponding frame bounds. This concept will play a crucial role in the continuous "time"-variables.

In the case of $a_{\min} = a_{\max}$, the family $g_1, \ldots, g_n$ is called a *tight frame*, which behaves almost like an orthogonal basis though it might be linearly dependent. Every frame $g_1, \ldots, g_n$ can be converted into the tight frame $S^{-1/2}g_1, \ldots, S^{-1/2}g_n$, where $S^{-1/2}$ is the square root of the inverse of the so-called *frame operator* $S = AA'$. This square root can be obtained via the singular value decomposition of S by simply replacing the singular values by their inverse square root. A simple computation shows that for $A = U\Sigma V'$, the columns of the matrix UV' constitute the above described tight frame $S^{-1/2}g_1, \ldots, S^{-1/2}g_n$.

All these objects have an analogue description for continuous signals in L^2 as described in Section 4. But before, we discuss a particular choice of a basic system that enjoys a lot of structure which in turn induces fast numerical algorithms. In order to better understand this structure we have to describe the theory in a more abstract setting, which requires an extension of the underlying mathematical objects based on classical results in functional analysis.

3. Finite dimensional Gabor analysis

In the following we build a special family of basic vectors in $\mathbb{C}^m$ in order to decompose and recover signals of length m as described in the previous section. To this end we essentially take a single vector and derive the remaining basic vectors by regular cyclic shifts and modulations of this vector. In this way, we obtain a highly structured system.

First we introduce the basic notations. For convenience, a vector f in $\mathbb{C}^m$ is periodically extended on $\mathbb{Z}$ via

$$f(k + qm) = f(k)\,, \qquad q \in \mathbb{Z}\,.$$

The index set of a vector can be identified with the finite group $Z_m =$

$\{1,\dots,m\}$ and all indices exceeding m are understood modulo m (without explicit notation). In this sense, cyclic shifts are just another form of "ordinary shifts" on periodic functions.

The (index) *translation operator* T_k that rotates the index of a vector by k and the (frequency) *modulation operator* M_l that performs a frequency shift by l have the following matrix entries with respect to the standard basis in $\mathbb{C}^m$:

$$(T_k)_{uv} \;=\; \delta_{u+k,v} \quad \text{and} \quad (M_l)_{uv} \;=\; e^{2\pi i(u-1)l/m}\delta_{u,v}\,,$$

where δ denotes the Kronecker symbol.

By simply applying the product of these matrices to an arbitrary vector $f = (f(1),\dots,f(m))^t$ we see that they do not commute in general:

$$M_lT_kf = \big(f(k+1),\dots,w_m^{(m-k)l}f(m),w_m^{(m-k+1)l}f(1),\dots,w_m^{(m-1)l}f(k)\big)^t,$$

$$T_kM_lf = \big(w_m^{kl}f(k+1),\dots,w_m^{(m-1)l}f(m),f(1),\dots,w_m^{(k-1)l}f(k)\big)^t,$$

where $w_m = e^{2\pi i/m}$. We use the symbol ^{t} to indicate column vectors.

For a more compact description, we write for the time-frequency shift (first a time-shift is applied, and then the modulation corresponding to frequency shift):

$$\pi(\lambda) = M_lT_k \quad \text{with} \quad \lambda = (k,l) \in Z_m \times Z_m\,.$$

Note that $\pi(\lambda)$ are unitary matrices, i.e., $\pi(\lambda)^{-1} = \pi(\lambda)'$. From the relation between M_lT_k and T_kM_l, we can easily derive the commutation rule

$$\pi(\lambda_1)\pi(\lambda_2) = e^{2\pi i(l_1k_2-k_1l_2)}\pi(\lambda_2)\pi(\lambda_1)\,. \tag{5}$$

Because of (5), the set of time-frequency shift matrices do not constitute a (multiplicative) group. However, by taking into account this commutation property, time-frequency shift operations can be extended in such a way that they finally constitute a group, namely the so-called *Heisenberg group* [25].

Next we introduce the announced basic system whose structure is heavily based on this commutation rule. Given a lattice (subgroup) Λ in $Z_m \times Z_m$ and a vector g in $\mathbb{C}^m$, we say that the family $\{g_\lambda\}_{\lambda\in\Lambda}$ defined by

$$g_\lambda = \pi(\lambda)g, \qquad \lambda \in \Lambda,$$

is a (discrete) Gabor frame if it spans all $\mathbb{C}^m$. The associated frame matrix is the positive definite matrix S given by

$$Sf = GG'f = \sum_{\lambda\in\Lambda} \langle f, g_\lambda \rangle g_\lambda \, .$$

Every Gabor frame induces a dual frame as described in the previous section. For Gabor frames, however, the computation of the canonical dual can be reduced to the solution of a single linear equation of the form

$$Sh \;=\; g \tag{6}$$

because then the dual frame is simply given by $\{\pi(\lambda)h\}_{\lambda\in\Lambda}$ which makes Gabor frames so attractive for applications. This statement follows easily from the observation that the Gabor frame matrix commutes with all time-frequency shifts $\pi(\lambda)$, $\lambda \in \Lambda$, (and so does S^{-1}) since Λ is supposed to be a group and the factor in (5) simply drops out. What we obtain is the following Gabor representation of a signal $f \in \mathbb{C}^m$:

$$f \;=\; \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)h \rangle \pi(\lambda) g \;=\; \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda) h \, . \tag{7}$$

The Gabor representation shows also how we can reconstruct a signal from the coefficient vector $\langle f, \pi(\lambda)g\rangle$, $\lambda \in \Lambda$, which corresponds to the short-time Fourier transform (STFT) $V_g f$ of f with respect to the window g, short

$$V_g f(k,l) = \langle f, \pi(k,l) g \rangle = \sum_{j=1}^{m} f(j) \overline{g(j+k)} e^{-2\pi i j l/m} \, ,$$

sampled on the lattice Λ. This leads to the dual perspective in which we ask how we can recover f from samples of its STFT. As seen in the previous section, the frame approach of (7) already gives a satisfactory answer to the dual problem.

Frames, and in particular Gabor frames, are in general linearly dependent, which basically means that not all coefficient information is needed for completely recovering the signal. Indeed, if, for example, one sample of the sampled short-time Fourier transform is missing, say the one indexed by λ_o, we can still reconstruct f since $\{\pi(\lambda)g\}_{\lambda\in\lambda\in\Lambda\setminus\lambda_o}$ is a frame because of the underlying group structure. In this case, however, the dual frame can not be expected to be derived from a single vector.

Equation (6) reveals that one of the most important objects in Gabor analysis is the frame matrix S. Indeed, many studies in this field have been devoted to this operator not only for a better understanding of Gabor systems but also for designing fast numerical inversion schemes. In the last part of this section we give a small insight to fundamental results of the Gabor frame matrix.

We start with the special case of so-called separable lattices of the form $\Lambda = \alpha Z_m \times \beta Z_m$ where α and β are divisors of m. We define $\tilde{\alpha} = m/\alpha$ and $\tilde{\beta} = m/\beta$. (The case that $\alpha\beta$ divides m corresponds to *integer oversampling.*) Due to the fact that $\sum_{m=0}^{\tilde{\beta}-1} e^{2\pi i j m\beta/m} = 0$ if j does not divide $\tilde{\beta}$, the jl-th element of the frame matrix S is simply given by

$$(S)_{jl} = \begin{cases} \tilde{\beta} \sum_{n=0}^{\tilde{\alpha}-1} g(j-\alpha n)\overline{g(l-\alpha n)} & \text{if } |j-l| \text{ is divided by } \tilde{\beta} \\ 0 & \text{otherwise}\,, \end{cases} \tag{8}$$

which is called the *Walnut representation* of S for the discrete case [36]. The discrete Walnut representation implies the following properties of S:

(1) Only every $\tilde{\beta}$-th subdiagonal of S is non-zero.
(2) Entries along a subdiagonal are α-periodic.
(3) S is a block circulant matrix of the form

$$S = \begin{bmatrix} A_0 & A_1 & \dots & A_{\tilde{\alpha}-1} \\ A_{\tilde{\alpha}-1} & A_0 & \dots & A_{\tilde{\alpha}-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \dots & A_0 \end{bmatrix}$$

where A_s are non-circulant $\alpha \times \alpha$ matrices, with

$$(A_s)_{j,l} = (S)_{j+s\alpha,l+s\alpha} \tag{9}$$

for $s = 0, 1, \dots, \tilde{\alpha}-1$ and $j, l = 0, 1, \dots, \alpha-1$, [41].

This special case applies merely for separable lattices. In general, however, we have another powerful representation of the frame matrix, the so-called *Janssen representation.*

For a better understanding of the Janssen representation in finite dimension we introduce the Frobenius norm for $m \times n$ matrices

$$\|A\|_{\mathrm{Fro}} = \Big(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\Big)^{1/2} = \sqrt{\mathrm{tr}(A'A)}$$

where the *trace* $\mathrm{tr}(B)$ of B is the sum of its diagonal entries. (The Frobenius norm corresponds to the Hilbert-Schmidt norm in infinite dimension.) The

Frobenius norm can be derived from the inner product

$$\langle A, B\rangle_{\mathrm{Fro}} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\bar{b}_{ij}\,. \tag{10}$$

It can easily be seen that the matrix family $\{\pi(\lambda)\}_{\lambda\in Z_m\times Z_m}$ of all possible time-frequency shifts on $\mathbb{C}^m$ constitutes an orthogonal system with respect to $\langle\cdot,\cdot\rangle_{Fro}$ in the space of all complex valued $m\times m$ matrices, denoted by $\mathcal{M}(m)$. The system becomes orthonormal for the inner product

$$\langle A, B\rangle_F = \frac{1}{m}\langle A, B\rangle_{\mathrm{Fro}}\,.$$

As a consequence, every matrix in $(\mathcal{M}(m), \langle\cdot,\cdot\rangle_F)$ has a unique expansion (time-frequency representation) with respect to the orthonormal system $\{\pi(\lambda)\}_{\lambda\in Z_m\times Z_m}$ (the coefficients corresponding to the spreading function).

Also the frame matrix S of a Gabor frame can be decomposed with respect to all time-frequency shift matrices. This representation can be simplified due to the special structure of S as we will see in the following.

We define the *adjoint lattice* Λ° of Λ to consist of all tuples of elements in $Z_m\times Z_m$ for which

$$\pi(\lambda)\pi(\lambda^\circ) \;=\; \pi(\lambda^\circ)\pi(\lambda) \tag{11}$$

holds. Remark that Λ° is indeed an (additive) subgroup (lattice) of Z_m. In particular, according to (5), we must have

$$e^{2\pi i(kl^\circ - k^\circ l)/m} \;=\; 1 \tag{12}$$

for all $\lambda=(k,l)$ and $\lambda^\circ=(k^\circ,l^\circ)$.

We observe that the Gabor frame matrix S commutes with all time-frequency shifts $\pi(\lambda)$ with $\lambda\in\Lambda$. Combining the commutation rule (5) with the uniqueness of the time-frequency representation immediately induces, that the only coefficients that might be different from zero are those related to the dual lattice Λ°. Therefore, we have

$$S \;=\; \sum_{\lambda^\circ\in\Lambda^\circ} c_{\lambda^\circ}\pi(\lambda^\circ) \tag{13}$$

which is the so-called Janssen representation of S. The coefficients are actually given by the restriction of the short time Fourier transform to the dual lattice, i.e.,

$$c_{\lambda^\circ} \;=\; \langle S, \pi(\lambda^\circ)\rangle_F \;=\; \frac{n}{m}\langle g, \pi(\lambda^\circ)g\rangle \;=\; \frac{n}{m}V_g g(\lambda^\circ) \tag{14}$$

where n denotes the order of Λ. In order to derive the relation between the coefficients in (13) and the (discrete) STFT $V_g(g)$ we note that

$$(S)_{jk} = (GG')_{jk} = \sum_{\lambda\in\Lambda} e^{2\pi i(j-k)s_\lambda/m} g(j+r_\lambda)\overline{g}(k+r_\lambda)$$

where $\lambda = (r_\lambda, s_\lambda)$ and

$$(\pi(\lambda^\circ))_{jk} = e^{2\pi i(j-1)s^\circ/m}\delta_{j+r^\circ,k} \, .$$

Now we compute

$$\langle S, \pi(\lambda^\circ)\rangle_F = \frac{1}{m}\sum_{j=1}^{m}\sum_{k=1}^{m}(S)_{jk}\overline{(\pi(\lambda^\circ))}_{jk} =$$

$$\frac{1}{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\Big(\sum_{\lambda\in\Lambda} e^{2\pi i(j-k)s_\lambda/m} g(j+r_\lambda)\overline{g}(k+r_\lambda)\Big) e^{-2\pi i(j-1)s^\circ/m}\delta_{j+r^\circ,k} =$$

$$\frac{1}{m}\sum_{j=1}^{m}\sum_{\lambda\in\Lambda} e^{-2\pi i r^\circ s_\lambda/m} g(j+r_\lambda)\overline{g}(j+r^\circ+r_\lambda) e^{-2\pi i(j-1)s^\circ/m} =$$

$$\frac{1}{m}\sum_{\lambda\in\Lambda}\underbrace{e^{2\pi i(r_\lambda s^\circ - r^\circ s_\lambda)/m}}_{\overset{(12)}{=}1}\sum_{j=1}^{m} g(j)\overline{g}(j+r^\circ) e^{-2\pi i(j-1)s^\circ/m} =$$

$$\frac{n}{m}\sum_{j=1}^{m} g(j)\overline{\big(\pi(\lambda^\circ)g\big)}(j) = \frac{n}{m}\langle g, \pi(\lambda^\circ)g\rangle \, .$$

The coefficients $(c_{\lambda^\circ})_{\lambda^\circ\in\Lambda^\circ}$ are called the *Janssen coefficients* of S.

Relation (14) gives rise to efficient computations of the so-called Janssen coefficients by using the standard fast Fourier transform for the STFT.

The coefficients of the time-frequency expansion correspond to the so-called *spreading function* which will be introduced later in the continuous framework and has revealed many new insights into the study of Gabor analysis.

The ratio n/m denotes the redundancy of the Gabor frame. In case of maximal redundancy m, i.e., $\Lambda = Z_m \times Z_m$, the Gabor frame operator reduces to a multiple of the identity, according to the Janssen representation, and corresponds to a tight frame in which the synthesis Gabor atom coincides with the canonical analysis atom. This situation simply reflects the full STFT and its inversion formula in analogy to the continuous case.

Having two Gabor systems $\{g_\lambda\}$ and $\{\gamma_\lambda\}$, the corresponding frame type operator $S_{\gamma,g} = \Gamma G'$ does also commute with all $\pi(\lambda)$ and we again have

$$S_{\gamma,g} \;=\; \sum_{\lambda^\circ \in \Lambda^\circ} c_{\lambda^\circ} \pi(\lambda^\circ)$$

with $c_{\lambda^\circ} = \frac{n}{m} V_\gamma(g)(\lambda^\circ) = \frac{n}{m} \langle \gamma, \pi(\lambda^\circ) g\rangle$ which implies that g and γ are dual frames if and only if $\langle \gamma, \pi(\lambda^\circ) g\rangle = \delta_{0,\lambda^\circ}$, $\lambda^\circ \in \Lambda^\circ$ (*Wexler-Raz identity*).

All the stated results have an analogue description for continuous signals (where the signal space is infinite dimensional). However, convergence problems (not occurring in finite dimension) require a careful treatise of this subject. In our next section we discuss basic functional analytic principles that allow to generalize the Gabor concept to continuous signals mainly by controlling convergence issues.

4. Frames and Riesz bases

Let $\{g_k\}_{k\in\mathbb{Z}}$ be a family in an infinite dimensional (separable) Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. The classical examples are the Hilbert space L^2 of (equivalence classes of measurable) functions of finite energy (L^2-norm) and the sequence space ℓ^2 consisting of square summable complex-valued sequences. Similar to the finite dimensional case, we want to represent a signal f in $\mathcal{H}$ as a (now possibly infinite) linear combination of the form

$$f \;\sim\; \sum_k c_k g_k \,.$$

At this point there a several questions that arise naturally when considering an infinite sum. First, by convention, we want the sum to converge in the (prescribed) Hilbert-norm, i.e., $\lim_{K\to\infty} \|f - \sum_{k=1}^{K} c_k g_k\| \longrightarrow 0$. Secondly, the sum should converge to the same limit (preferably f) regardless of the summation order we choose (known as *unconditional convergence*). A more subtle point is that we would like to have a continuous linear dependency between the signal f and the coefficients c_k in order to avoid pathological cases in which small alterations in the signal result in uncontrollable changes in the corresponding coefficient sequence and vice-versa. This technical detail accounts for numerical stability.

Obviously, all these requirements are trivially fulfilled in the finite dimensional case. For an infinite family $\{g_k\}$, however, these assumptions have to be ensured before dealing with decomposition and reconstruction issues. Fortunately, there exist concepts in functional analysis that do exactly fit this kind of requirements. For a precise description we need the following definitions.

Definition 1: A family $\{g_k\}$ of a Hilbert space $\mathcal{H}$ is complete in $\mathcal{H}$ if the set of finite linear combinations of $\{g_k\}$, write $\operatorname{span}(g_k)$, is dense in $\mathcal{H}$, i.e., every f in $\mathcal{H}$ can be arbitrarily well approximated by elements in $\operatorname{span}(g_k)$ with respect to the $\mathcal{H}$-norm.

In the mathematical literature complete systems are often called "total". The definition makes no claim about the "cost" of approximation. In other words, it is allowed to use more and more complicated coefficient sequences as the approximation quality is increased. In particular, total families do not necessarily allow a series expansion of arbitrary elements from the given Hilbert space.

Definition 2: The family $\{g_k\}$ of a Hilbert space $\mathcal{H}$ is a basis for $\mathcal{H}$ if for all $f \in \mathcal{H}$ there exists unique scalars $c_k(f)$ such that

$$f \;=\; \sum_k c_k(f) g_k \,.$$

In contrast to complete sequences, a basis always induces a series expansion.

Definition 3: The sequence $\{g_k\}$ in a Hilbert space $\mathcal{H}$ is called a Bessel sequence if

$$\sum_k |\langle f, g_k\rangle|^2 < \infty \,, \qquad f \in \mathcal{H} \,.$$

Definition 4: A family $\{g_k\}$ of a Hilbert space $\mathcal{H}$ is a Riesz sequence if there exist bounds $A, B > 0$ such that

$$A\|c\|_{\ell^2}^2 \leq \Big\| \sum_k c_k g_k \Big\|^2 \leq B\|c\|_{\ell^2}^2 \,, \qquad c \in \ell^2 \,.$$

Riesz sequences preserve many properties of orthonormal sets [5]. A Riesz sequence which generates all $\mathcal{H}$ is called a *Riesz basis* for $\mathcal{H}$. Riesz bases are somehow "distorted" orthonormal bases as described in the following lemma which reveals all useful properties of a Riesz basis [27].

Lemma 5: *Let $\{g_k\}$ be a sequence in a Hilbert space $\mathcal{H}$. The following are equivalent.*

(1) *$\{g_k\}$ is a Riesz basis for $\mathcal{H}$.*
(2) *$\{g_k\}$ is an unconditional basis for $\mathcal{H}$ and g_k are uniformly bounded.*
(3) *$\{g_k\}$ is a basis for $\mathcal{H}$, and $\sum_k c_k g_k$ converges if and only if $\sum_k |c_k|^2$ converges.*
(4) *There is an equivalent inner product on $\mathcal{H}$ for which $\{g_k\}$ is an orthonormal basis for $\mathcal{H}$.*
(5) *$\{g_k\}$ is a complete Bessel sequence and possesses a bi-orthogonal system $\{h_k\}$ that is also a complete Bessel sequence.*

The last item of the lemma says that there exists a unique sequence $\{h_k\}$ such that $\langle g_k, h_j \rangle = \delta_{kj}$ which, combined with the second statement, induces the representation

$$f \;=\; \sum_k \langle f, h_k \rangle g_k \;=\; \sum_k \langle f, g_k \rangle h_k\,, \qquad f \in \mathcal{H}\,.$$

Hence, Riesz bases are potential candidates for our purpose of signal representation. We point out that the coefficient sequence is always square summable which is an important stability criterion. In the next section we will give an example of a basis which is not stable, i.e., the coefficient sequence is not summable at all for special examples.

So far, the systems we are considering allow only unique expansions with respect to the coefficients. In applications it is sometimes more useful to weaken this property. This can be obtained by looking for overcomplete (linearly dependent) sets which is implemented in the concept of frames introduced by Duffin and Schaeffer in 1952 [11].

Definition 6: A family $\{g_k\}$ of a Hilbert space $\mathcal{H}$ is a frame of $\mathcal{H}$ if there exist bounds $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{k\in\mathbb{Z}} |\langle f, g_k \rangle|^2 \le B\|f\|^2\,, \qquad f \in \mathcal{H}\,. \tag{15}$$

If $A = B$, then $\{g_k\}_{k\in\mathbb{Z}}$ is called a tight frame.

The synthesis map $G : \ell^2 \to \mathcal{H}$ of a frame $\{g_k\}$ is defined by

$$G : (c_k) \to \sum_k c_k g_k \,.$$

Its adjoint G^* is the analysis operator $G^* f = (\langle f, g_k \rangle)$. The *frame operator* S is defined by

$$Sf = GG^* f = \sum_{k \in \mathbb{Z}} \langle f, g_k \rangle g_k \,, \qquad f \in \mathcal{H} \,.$$

By (15), the frame operator satisfies

$$A\langle f, f \rangle \leq \langle Sf, f \rangle \leq B\langle f, f \rangle \,, \qquad f \in \mathcal{H} \,,$$

and is, therefore, bounded, positive, and invertible. The inverse operator S^{-1} is obviously also positive and has therefore a square root $S^{-1/2}$ (self-adjoint), [39]. It follows that the sequence $\{S^{-1/2} g_k\}$ is a tight frame with $A = B = 1$.

Every orthonormal basis of $\mathcal{H}$ is a Riesz basis of $\mathcal{H}$ and every Riesz basis of $\mathcal{H}$ is also a frame. The important difference between a Riesz basis and a frame is that the null space $\mathcal{N}(G)$ of the synthesis map G of a frame $\{g_k\}$ is in general non-trivial, which is equivalent to the statement that the range of the analysis map G^* is a (closed) proper subspace of ℓ^2.

The sequence $\{\tilde{g}_k\}$ with $\tilde{g}_k = S^{-1} g_k$, is also a frame with frame bounds $1/B$ and $1/A$. It is a dual frame for $\{g_k\}$ in the sense that

$$f \;=\; \sum_k \langle f, \tilde{g}_k \rangle g_k \;=\; \sum_k \langle f, g_k \rangle \tilde{g}_k \,, \qquad f \in \mathcal{H} \,.$$

Again, we see that frames do indeed fit our purpose for signal analysis and signal recovery. In contrast to Riesz bases, frames have, in general, no bi-orthogonal relation. Moreover, the dual frame is not unique. The canonical dual $\{S^{-1} g_k\}$ is the one that produces minimal ℓ^2 coefficients as already shown in [11]. It corresponds to the pseudo-inverse of the analysis operator in finite dimension. For alternative dual frames there exist constructive approaches that rely on the canonical dual. In [6,32], it is shown that any dual frame of $\{g_k\}$ can be written as

$$S^{-1} g_k + h_k - \sum_j \langle S^{-1} g_k, g_j \rangle h_j \,, \tag{16}$$

where $\{h_k\}$ is a Bessel sequence.

The lack of uniqueness has the advantage that if one coefficient is missing out of the sequence $\langle f, g_k \rangle$, the whole signal can still be completely recovered as long as $\{g_k\}$ is a frame but no Riesz basis. Similarly, any frame that is not a Riesz basis is still a frame when discarding single frame elements. Studies about the conservation of the frame property when discarding frame elements are known as *excesses of frames* [1,2].

5. Gabor analysis on $\mathbf{L^2}$

We define the Fourier transform of an integrable function by $\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t)e^{-2\pi it\omega}dt$. For an element $\lambda = (x, \omega) \in \mathbb{R}^{2d}$ we define the time-frequency shift $\pi(\lambda)$ by

$$\pi(\lambda) = M_\omega T_x$$

where T_x is the translation operator $T_x f = f(\cdot - x)$ and M_ω is the modulation operator $M_\omega f = e^{2\pi i\omega\cdot} f(\cdot)$. In analogy to the finite dimensional model, note that

$$\pi(\lambda_2)\pi(\lambda_1) = e^{2\pi i(x_1\omega_2 - x_2\omega_1)}\pi(\lambda_1)\pi(\lambda_2)$$

for $\lambda_1 = (x_1, \omega_1), \lambda_2 = (x_2, \omega_2) \in \mathbb{R}^{2d}$.

A time-frequency lattice Λ is a discrete subgroup of $\mathbb{R}^{2d}$ $(= \mathbb{R}^d \times \hat{\mathbb{R}}^d)$ with compact quotient. Its redundancy $|\Lambda|$ is the reciprocal value of the measure of a fundamental domain for the quotient $\mathbb{R}^{2d}/\Lambda$.

For a lattice Λ in $\mathbb{R}^{2d}$ and a so-called *Gabor atom* $g \in L^2$ we define the associated Gabor family by

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g\}_{\lambda\in\Lambda} .$$

If $\mathcal{G}(g, \Lambda)$ is a frame for L^2, we call it a *Gabor frame*. Since Λ has a group structure, the frame operator

$$Sf = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g$$

has the property that it commutes with all time-frequency shifts of the form $\pi(\lambda)$ for $\lambda \in \Lambda$. Therefore, the canonical dual frame of $\mathcal{G}(g, \Lambda)$ is simply given by $\mathcal{G}(h, \Lambda)$ with $h = S^{-1}g$. The fact that a canonical dual frame of a Gabor frame is again a Gabor frame, i.e., generated by a single function, is the key property in many applications. It reduces computational issues to solving the linear system $Sh = g$.

A special and widely studied case are separable lattices of the form $\alpha\mathbb{Z} \times \beta\mathbb{Z}$ for some positive lattice parameters α and β, whose redundancy is simply $(\alpha\beta)^{-1}$. The prototype of a function generating Gabor frames for such separable lattices is the Gaussian

$$\psi(x) = e^{-\pi x^2 \sigma^2} . \tag{17}$$

for some real $\sigma > 0$. The Gaussian generates a Gabor frame if and only if $\alpha\beta < 1$ [35,40]. We emphasize that for $\alpha\beta = 1$ the Gaussian generates an *unstable* generating system for L^2, i.e., the resulting Gabor family is complete but coefficient sequences must not be bounded. In this context we mention a central result, the so-called *density theorem* and refer to [25] for detailed discussions. An elegant elementary proof of the density theorem has been provided by Janssen [29].

Theorem 7: Assume that $\mathcal{G}(g, \alpha, \beta)$ is a frame. Then, $\alpha\beta \leq 1$. Moreover, $\mathcal{G}(g, \alpha, \beta)$ is a Riesz basis for L^2 if and only if $\alpha\beta = 1$.

In his seminal paper [20], Gabor chose the integer lattice $\alpha = \beta = 1$ in $\mathbb{R}^2$ and used the Gaussian in order to define a Gabor system with maximal time-frequency localization. However, as mentioned above, this system is no longer stable though complete, and, indeed, the celebrated Balian-Low Theorem [3,34] states that good time-frequency localization and Gabor Riesz bases are not compatible:

Theorem 8: (Balian-Low) If $\mathcal{G}(g, 1, 1)$ constitutes a Riesz basis for $L^2(\mathbb{R})$, then

$$\int_{\mathbb{R}} |g(t)|^2 t^2 dt \int_{\mathbb{R}} |\hat{g}(\omega)|^2 \omega^2 d\omega = \infty .$$

The Balian-Low Theorem reveals a form of uncertainty principle and has inspired fundamental research, see [25] and references therein.

In the sequel we state some fundamental results on Gabor frames and the Gabor frame operator (Gabor frame-type operator). To this end we need the notion of the *adjoint lattice* Λ° of Λ which is, similarly to the discrete case, the set of all elements in $\mathbb{R}^{2d}$ that satisfy the commutation property

$$\pi(\lambda^\circ)\pi(\lambda) = \pi(\lambda)\pi(\lambda^\circ) \quad \text{for all} \quad \lambda \in \Lambda .$$

Note that Λ° is again a lattice of $\mathbb{R}^{2d}$ (and that $\Lambda^{\circ\circ} = \Lambda$). Instead of the frame operator we will use the more general notion of a frame-type operator $S_{g,\gamma,\Lambda}$ associated to the pair (g, γ), where γ takes the role of an "analyzing" and g the role of a "synthesizing" window:

$$S_{g,\gamma,\Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma\rangle\, \pi(\lambda) g\,, \qquad f \in L^2\,.$$

This sum converges in L^2 for all $f \in L^2$ as long as both functions g, γ are Bessel atoms for Λ, that is, $\mathcal{G}(g, \Lambda)$ and $\mathcal{G}(\gamma, \Lambda)$ are Gabor Bessel families. For the fundamental results to hold with respect to norm convergence we need a little bit more than Bessel sequences, namely, that both atoms g and γ satisfy

$$\sum_{\lambda^\circ \in \Lambda^\circ} |\langle g, g_{\lambda^\circ}\rangle| < \infty\,. \tag{A'}$$

A' is also known as the *Tolimieri-Orr's condition.* This somewhat technical property is used for controlling convergence problems (by altering the convergence definition Condition A' can be weakened). Condition A' is in general not easy to verify. In particular, if Condition A' holds for one lattice, there is, in general, no guarantee that it holds also for a different lattice. This problem, however, is overcome by the Feichtinger algebra S_0 which defines a class of functions for which Condition A' is satisfied for any lattice in $\mathbb{R}^{2d}$. We introduce this algebra in Section 7.

We summarize the fundamental results of Gabor analysis in the following theorem that is given in [18] in a slightly more general context. The statements go back to the seminal papers [9,28,42]. They, however, are all consequences of the fundamental identity of Gabor analysis extensively studied in [16].

Theorem 9: Let Λ be a lattice in $\mathbb{R}^{2d}$ with adjoint lattice Λ°. Then, for g, h satisfying A', the following hold.

(1) (Fundamental Identity of Gabor Analysis)

$$\sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma\rangle \langle \pi(\lambda) g, h\rangle = |\Lambda| \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, \pi(\lambda^\circ)\gamma\rangle\, \langle \pi(\lambda^\circ) f, h\rangle \tag{18}$$

for all $f, h \in L^2$, where both sides converge absolutely.

(2) (Wexler-Raz Identity)

$$S_{g,\gamma,\Lambda} f = |\Lambda| \cdot S_{f,\gamma,\Lambda^\circ} g \tag{19}$$

for all $f \in L^2$.

(3) (Janssen Representation)

$$S_{g,\gamma,\Lambda} = |\Lambda| \sum_{\lambda^\circ \in \Lambda^\circ} \langle \gamma, \pi(\lambda^\circ) g \rangle \, \pi(\lambda^\circ) \tag{20}$$

where the series converges unconditionally in the strong operator sense.

In Section 8 we explicitly derive the Janssen representation of the Gabor frame operator from advanced concepts in harmonic analysis and provide a much deeper insight into this topic.

Another important result is the Ron-Shen Duality Principle which is often referred to [38] although it appeared already in [28] and [9] and was announced in [37].

Theorem 10: Let $g \in L^2$ and Λ be a lattice in $\mathbb{R}^{2d}$ with adjoint Λ°. Then the Gabor system $\mathcal{G}(g,\Lambda)$ is a frame for L^2 if and only if $\mathcal{G}(g,\Lambda^\circ)$ is a Riesz basis for its closed linear span. In this case, the quotient of the two frame bounds and quotient of the Riesz bounds (alternatively the condition number of the corresponding frame operator and the Gramian matrix, respectively), coincide.

The last important identity in Gabor Analysis that we want to present in this section is the Wexler-Raz Biorthogonality Relation which basically says that g and γ are dual Gabor windows if and only if $S_{g,\gamma,\Lambda} = Id$. That is, according to Janssen Representation, exactly the case when

$$\langle \gamma, \pi(\lambda^\circ) g \rangle \;=\; |\Lambda|^{-1} \, \delta_{0,\lambda^\circ} \,.$$

Alternatively, this relation can be described by what is a true biorthogonality (using again Kronecker's Delta):

$$\langle \pi(\lambda^{\circ\prime})\gamma, \pi(\lambda^\circ) g \rangle \;=\; |\Lambda|^{-1} \, \delta_{\lambda^{\circ\prime},\lambda^\circ} \,.$$

So far we have seen that, similar to the finite dimensional model, the Gabor frame operator in the continuous case plays a central role in Gabor theory. Indeed, it is the key object that allows for opening different perspectives, and bridges Gabor analysis to other research areas. In the next section we describe basic and more advanced studies in harmonic analysis that contribute to a better understanding of the Gabor frame operator.

6. Time-frequency representations

Traditionally we extract the frequency information of a signal f by means of the Fourier transform $\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t)e^{-2\pi it\omega}dt$. If we know $\hat{f}(\omega)$ for all frequencies ω, then our signal f can be reconstructed by the inversion formula $f(t) = \int_{\widehat{\mathbb{R}^d}} \hat{f}(\omega)e^{2\pi it\omega}d\omega$ (valid pointwise or in the quadratic mean).

However, in many situations it is of relevance to know, *how long* each frequency appears in the signal f, e.g., for a pianist playing a piece of music. Mathematically this leads to the study of functions $S(f)(t,\omega)$ of the signal f, which describe the time-frequency content of f over "time" t. In the following we mention the most prominent time-frequency representations.

In the last century researchers such as E. Wigner, Kirkwood, and Rihaczek had invented different time-frequency representations, [43,31]. The work of Wigner and Kirkwood was motivated by the description of a particle in quantum mechanics by a joint probability distribution of position and momentum of the particle. More concretely, in 1932 Wigner introduced the first time-frequency representation of a function $f \in L^2(\mathbb{R}^d)$ by

$$W(f)(x,\omega) = \int_{\mathbb{R}^d} f(x+\frac{t}{2})\overline{f(x-\frac{t}{2})}e^{-2\pi i\omega t}dt,$$

the so called *Wigner distribution* of f. Later Kirkwood proposed another time-frequency representation, which was in a different context rediscovered by Rihaczek. Both researchers associated to a function $f \in L^2(\mathbb{R}^d)$ the following expression

$$R(f)(x,\omega) = f(x)\overline{\hat{f}(\omega)}e^{-2\pi ix\omega},$$

the *Kirkwood-Rihaczek distribution* of f.

Nowadays, the *short-time Fourier transform* (STFT) has become the standard tool for (linear) time-frequency analysis. It is used as a measure of the time-frequency content of a signal f (energy distribution), but it also establishes a connection to the Heisenberg group.

The STFT provides information about local (smoothness) properties of the signal f. This is achieved by localization of f near t through multiplication with some *window function* g and subsequently applying the Fourier transform providing information about the frequency content of f in this segment. Typically g is concentrated around the origin. If g is compactly supported only a segment of f in some interval or ball around t is relevant, but g can be any non-zero Schwartz function such as the Gaussian. Overall we have:

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\omega}dt, \quad \text{for} \quad (x,\omega) \in \mathbb{R}^{2d}. \tag{21}$$

Before we discuss the properties of the STFT we recall the key players of our game: *translation* T_x, *modulation* M_ω, and *time-frequency shifts* $\pi(x,\omega) = M_\omega T_x$ of a signal f already introduced in Section 5.

In 1927 Weyl pointed out that the translation and modulation operator satisfy the following commutation relation

$$T_x M_\omega = e^{-2\pi i x\omega} M_\omega T_x, \quad (x,\omega) \in \mathbb{R}^{2d}. \tag{22}$$

$\{T_x : x \in \mathbb{R}^d\}$ and $\{M_\omega : \omega \in \mathbb{R}^d\}$ are Abelian groups of unitary operators, with the infinitesimal generators given by differentiation and multiplication operator, respectively. Therefore the commutation relation (22) is the analogue of Heisenberg's commutation relation for the differentiation and multiplication operator.

The time-frequency shifts $M_\omega T_x$ for $(x,\omega) \in \mathbb{R}^{2d}$ satisfy the following composition law:

$$\pi(x,\omega)\pi(y,\eta) = e^{-2\pi i x\cdot\eta}\pi(x+y,\omega+\eta), \tag{23}$$

for $(x,\omega),(y,\eta)$ in the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$. i.e. the mapping $(x,\omega) \mapsto \pi(x,\omega)$ defines (only) a *projective representation* of the time-frequency plane (viewed as an Abelian group) $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$. By adding a toral component, i.e. $\tau \in \mathbb{C}$ with $|\tau| = 1$ one can augment the phase space $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ to the so-called *Heisenberg group* $\mathbb{R}^d \times \widehat{\mathbb{R}^d} \times T$ and the mapping $(x,\omega,\tau) \mapsto \tau M_\omega T_x$ defines a (true) unitary representation of the Heisenberg group [19], the so-called *Schrödinger representation*. From this point of view the definition of $V_g f$ can be interpreted as representation coefficients:

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle, \qquad f,g \in L^2(\mathbb{R}^d).$$

The STFT is linear in f and conjugate linear in g. The choice of the window function g influences the properties of the STFT remarkably. One example of a good window class is the Schwartz space of rapidly decreasing functions. Later we will discuss another function space, which is perfectly suited as a good class of windows, *Feichtinger's algebra*.

Furthermore, for $f, g \in L^2(\mathbb{R}^d)$ the STFT $V_g f$ is uniformly continuous on $\mathbb{R}^{2d}$, i.e., we can sample the $V_g f$ without a problem. This fact is of great relevance in the discussion of Gabor frames.

By Parseval's theorem and an application of the commutation relations (22) we derive the following relation

$$V_g f(x,\omega) = e^{-2\pi i x\omega} V_{\hat{g}} \hat{f}(\omega, -x), \tag{24}$$

which is sometimes called the *fundamental identity of time-frequency analysis* [25]. The equation (24) expresses the fact that the STFT is a joint time-frequency representation and that the Fourier transform amounts to a rotation of the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ by an angle of $\frac{\pi}{2}$ whenever the window g is Fourier invariant. Another important consequence of the definition of STFT (21) and the commutation relations (22) is the *covariance property* of the STFT:

$$V_g(T_u M_\eta f)(x,\omega) = e^{-2\pi i u\omega} V_g f(x-u, \omega-\eta). \tag{25}$$

Later we will draw an important conclusion of the basic identity of time-frequency analysis (24) and the covariance property of the STFT (25): isometric Fourier invariance and the invariance under TF-shifts of Feichtinger's algebra.

As for the Fourier transform there is also a Parseval's equation for the STFT which is referred to as *Moyal's formula*.

Lemma 11: *(Moyal's Formula) Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$ then $V_{g_1} f_1$ and $V_{g_2} f_2$ are in $L^2(\mathbb{R}^{2d})$ and the following identity holds:*

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle}. \tag{26}$$

Moyal's formula implies that orthogonality of windows g_1, g_2 resp. of signals f_1, f_2 implies orthogonality of their STFT's. Most importantly, we observe that for normalized $g \in L^2(\mathbb{R}^d)$ (i.e., with $\|g\|_2 = 1$) one has:

$$\|V_g f\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2(\mathbb{R}^d)},$$

for all $f \in L^2(\mathbb{R}^d)$, i.e., the STFT is an isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$.

Another consequence of Moyal's formula is an inversion formula for the STFT. Assume that the analysis window $g \in L^2(\mathbb{R}^d)$ and the synthesis window $\gamma \in L^2(\mathbb{R}^d)$ satisfy $\langle g, \gamma \rangle \neq 0$. Then for $f \in L^2(\mathbb{R}^d)$

$$f = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^{2d}} \langle f, \pi(x,\omega) g \rangle \, \pi(x,\omega)\gamma \, dx d\gamma. \tag{27}$$

We observe that in contrast to the Fourier inversion the building blocks of the STFT inversion formula are just time-frequency shifts of a square-integrable function. Therefore also the Riemannian sums corresponding to this inversion integral are functions in $L^2(\mathbb{R}^d)$ and are even norm convergent in $L^2(\mathbb{R}^d)$ for nice windows (from Feichtinger's algebra, see later).

Recently the Heisenberg uncertainty principle, which expresses in many different ways that *a function f and its Fourier transform $\hat{f}$ cannot be both localized simultaneously*, has received much attention. We point the reader to the work of Gröchenig, Janssen, Hogan, Lakey, and others for results on uncertainty principles for time-frequency representations.

A first example is the following result of Lieb [33], which is derived from an application of Beckner's sharp Hausdorff-Young and Young's inequalities to $V_g f(x, \omega) = (\widehat{f \cdot T_x \overline{g}})(\omega)$:

Lemma 12: *If $f, g \in L^2(\mathbb{R}^d)$ and $1 \leq p < 2$, then*

$$\Big(\iint_{\mathbb{R}^{2d}} |V_g f(x, \omega)|^p dx d\omega \Big)^{1/p} \geq \Big(\frac{2}{p} \Big)^{d/p} \|f\|_2 \|g\|_2 \tag{28}$$

The inequality is reversed for $2 \leq p < \infty$.

Results of this kind can be taken as a motivation (although it was not the original one) for the introduction of function spaces characterized by summability and decay properties of the STFT of their elements. In 1983 Feichtinger introduced such a family of Banach spaces, the so-called *modulation spaces*. They have shown to be the right setup for a deeper understanding of operators in time-frequency analysis. In the sequel we will meet two members of the scale of modulation spaces: Feichtinger's algebra $S_0(\mathbb{R}^d)$ and its dual space $S_0'(\mathbb{R}^d)$. In this setup Lieb's inequality expresses just embeddings of certain modulation spaces into $L^2(\mathbb{R}^d)$. Gröchenig and some of his collaborators have extensively studied uncertainty principles as embeddings of certain weighted L^p-spaces into modulation spaces, [24].

7. The Gelfand triple $(S_0, L^2, S_0')(\mathbb{R}^d)$

Since Feichtinger's discovery of the Segal algebra $S_0(\mathbb{R}^d)$ in 1979, many results have shown that $S_0(\mathbb{R}^d)$ is a good substitute for Schwartz's space $\mathcal{S}(\mathbb{R}^d)$ of test functions (except if one is interested in a discussion of partial differential equations). Furthermore, $S_0(\mathbb{R}^d)$ has turned out to be the appropriate setting for the treatment of questions in harmonic analysis on $\mathbb{R}^d$ (actually on a general locally compact Abelian group G, even without using structure theory). In this section we recall well-known properties of $S_0(\mathbb{R}^d)$ which we will need later in our discussion of Gabor frame operators. Nowadays the space $S_0(\mathbb{R}^d)$ is called *Feichtinger's algebra* since it is a Banach algebra with respect to pointwise multiplication and convolution.

A function in $f \in L^2$ is (by definition) in the subspace $S_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{L^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is a Banach space, for any fixed, non-zero $g \in S_0(\mathbb{R}^d)$, and different windows g define the same space and equivalent norms. Since $S_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable windows. Often it is convenient to use the Gaussian as a window.

The above definition of $S_0(\mathbb{R}^d)$ (different from the original one) allows for an easy derivation of the basic properties of Feichtinger's algebra in the following lemma. Although they have appeared in various publications, cf. [25], we include the proofs as examples for the derivation of norm-estimates, as they are used in many different areas of time-frequency analysis.

Lemma 13: *Let $f \in S_0(\mathbb{R}^d)$, then the following holds:*

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.
(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

Proof:

(1) For $z = (u,\eta)$ in the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ one has:

$$\|\pi(u,\eta)f\|_{S_0} = \iint_{\mathbb{R}^{2d}} |V_g f(x-u, \omega-\eta)|\, dx d\omega =$$

$$= \iint_{\mathbb{R}^{2d}} |V_g f(x,\omega)|\, dx d\omega = C\|f\|_{S_0}.$$

(2) The key of the argument is an application of the fundamental identity of time-frequency analysis (24) to a Fourier invariant window g and the independence of the definition of $S_0(\mathbb{R}^d)$ from $g \in \mathcal{S}(\mathbb{R}^d)$. For simplicity we choose g the (Fourier invariant) Gaussian $g_0(x) = 2^{d/4}e^{-\pi x^2}$:

$$\|\hat{f}\|_{S_0} = \iint_{\mathbb{R}^{2d}} |V_{g_0}\hat{f}(x,\omega)|\, dx d\omega \;=\; \iint_{\mathbb{R}^{2d}} |V_{\widehat{g_0}}\hat{f}(x,\omega)|\, dx d\omega \;=$$

$$\iint_{\mathbb{R}^{2d}} |V_{g_0}f(-\omega,x)|\, dx d\omega \;=\; \iint_{\mathbb{R}^{2d}} |V_{g_0}f(x,\omega)|\, dx d\omega \;=\; \|f\|_{S_0}. \quad \square$$

Later we will need that $S_0(\mathbb{R}^d)$ is *dense* and *continuously embedded* into $L^p(\mathbb{R}^d)$ for any $p \in [1,\infty)$. The original motivation for Feichtinger's introduction of $S_0(\mathbb{R}^d)$ was the search for a *smallest* member in the family of all time-frequency homogenous Banach spaces. For a proof of all these assertions we refer the reader to the original paper of Feichtinger [F81] or Gröchenig's book on time-frequency analysis [25].

Another reason for the usefulness of $S_0(\mathbb{R}^d)$ is the fact that $S_0(\mathbb{R}^d)$ is a natural domain for the application of Poisson summation formula [24].

Lemma 14: *Let Λ be a lattice in $\mathbb{R}^d$ and $f \in S_0(\mathbb{R}^d)$ then*

$$\sum_{\lambda\in\Lambda} f(\lambda) \;=\; |\Lambda|^{-1} \sum_{\lambda^\perp\in\Lambda^\perp} \hat{f}(\lambda^\perp) \tag{29}$$

holds pointwise and with absolute convergence.

Here $\Lambda^\perp$ is the orthogonal lattice for Λ, e.g. $\Lambda^\perp = (A^{-1})^t\mathbb{Z}^d$ for $\Lambda = A\mathbb{Z}^d$, where A is a non-singular matrix describing Λ.

In 1958 I. M. Gelfand and A G. Kostyuchenko introduced Gelfand triples in their study of the spectral theory of self-adjoint operators [21]. They were motivated by the work of Dirac on the foundations of quantum mechanics and Schwartz's theory of distributions.

An important result of linear algebra is the theorem on the existence of eigenvectors for any self-adjoint linear operator A on $\mathbb{R}^d$. The situation changes drastically when one passes from the finite to the infinite-dimensional case, since it can happen that a unitary operators does not have any (non-zero) eigenvector. Particular examples of such operators are the translation operator T_x and the modulation operator M_ω on $L^2(\mathbb{R}^d)$.

Let us present an easy argument showing that the translation operator $T_x, x \neq 0$, has no eigenvectors in $L^2(\mathbb{R}^d)$. Assume that $f \in L^2(\mathbb{R}^d)$ satisfies

$$T_x f(t) = a f(t), \tag{30}$$

which by, the Fourier transform, is equivalent to

$$M_{-x}\hat{f}(\omega) = a\hat{f}(\omega) \quad \text{a.e..} \tag{31}$$

But this is only possible if the function $\hat{f}$ equals zero a.e., up to the points with $e^{2\pi i\omega x} \neq a$, i.e., it differs from zero only on a set of measure zero, hence $\hat{f} = 0$, and finally $f = 0 \in L^2(\mathbb{R}^d)$. In other words, the translation operator T_x does not have eigenvectors in the space $L^2(\mathbb{R}^d)$. On the other hand, we are not too far off with the claim that T_x has the eigenvectors $e^{-2\pi it\omega}$ corresponding to the eigenvalue $e^{2\pi ix\omega}$, and the claim that any function f in $L^2(\mathbb{R}^d)$ can be (kind of) expanded in terms of the eigenvectors $e^{-2\pi it\omega}$, by suitable interpretation of the inversion formula for the Fourier transform (valid pointwise for $f \in S_0(\mathbb{R}^d)$):

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) e^{2\pi it\omega} d\omega. \tag{32}$$

Furthermore, the action of the translation operator is given by

$$T_x f(t) = \int_{\mathbb{R}^d} e^{2\pi ix\omega} \hat{f}(\omega) e^{2\pi it\omega} d\omega,$$

which is a continuous analog of the spectral decomposition of a self-adjoint operator in $\mathbb{R}^d$.

More concretely, the system of eigenfunctions $\{e^{-2\pi it\omega} : \omega \in \widehat{\mathbb{R}^d}\}$ is complete in the sense that for any function f in $L^2(\mathbb{R}^d)$ Parseval's equality holds

$$\int_{\mathbb{R}^d} |f(t)|^2 dt = \int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 d\omega.$$

The obvious problem is the fact that $L^2(\mathbb{R}^d)$ does not contain the system of eigenvectors of the translation operator T_x. But they can be considered as linear functionals on $S_0(\mathbb{R}^d)$. This as well es several similar observations suggests to study operators on a Hilbert space via a dense subspace and its associated dual space. In our example it is actually possible to start from $S_0(\mathbb{R}^d)$ and construct $L^2(\mathbb{R}^d)$ as completion of $S_0(\mathbb{R}^d)$ with respect to norm corresponding to the usual scalar product $\langle f, g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t)}dt$.

In this context it turns out that $S_0(\mathbb{R}^d)$ has the important additional property that both δ-distributions and the pure frequencies $\chi_\omega(x) =$

$e^{-2\pi i x\omega}$ (for all $\omega \in \mathbb{R}^d$) are in a natural way elements of $S_0'(\mathbb{R}^d)$, i.e. define bounded linear functionals on $S_0(\mathbb{R}^d)$. As a consequence we are now in a situation similar to the one inspiring Gelfand to introduce what is nowadays called a Gelfand triple. The main idea is the observation, that a triple of spaces – consisting of the Hilbert space itself, a small (topological vector) space contained in the Hilbert space, and its dual – allows a much better description of the situation. The advantage in our case is the fact that we can even consider a Banach space, namely $S_0(\mathbb{R}^d)$. Hence we can work with the following formal definition:

Definition 15: A (Banach) *Gelfand triple* consists of some Banach space $(B, \|.\|_B)$, which is continuously and densely embedded into some Hilbert space $\mathcal{H}$, which in turn is w^*-continuously and densely embedded into the dual Banach space $(B', \|.\|')$.

We shall use the symbol $(B, \mathcal{H}, B')$ for such a triple of spaces. In this setting the inner product on $\mathcal{H}$ extends in a natural way to a pairing between B and B' (producing an anti-linear functional of the same norm).

As another consequence we mention an extension of an eigenvector of a bounded operator on a Hilbert space $\mathcal{H}$. Let A be a linear operator on a Banach space B. Then a linear functional F is a *generalized eigenvector* of A to the eigenvalue λ if

$$F(Af) = \lambda F(f), \qquad \text{for all} \quad f \in B.$$

This notion allows to interpret the characters $\chi_\omega(x) = e^{-2\pi i \omega x}$ as generalized eigenvectors for the translation operator T_x on $S_0(\mathbb{R}^d)$. Furthermore the set of generalized eigenvectors $\{\chi_\omega : \omega \in \mathbb{R}^d\}$ is complete by Plancherel's theorem, i.e., $\hat{f}(\omega) = \langle \chi_\omega, f \rangle = 0$ for all $\omega \in \mathbb{R}^d$, this implies $f \equiv 0$. This suggests to think of the Fourier transform of f at frequency ω as the evaluation of the linear functional $\langle \chi_\omega, f \rangle$.

The treatment of the translation operator T_x on $L^2(\mathbb{R}^d)$ is a particular case of a general theorem by Gelfand, that for any self-adjoint operator A on a Hilbert space $\mathcal{H}$ there exists a nuclear space and a complete system of generalized eigenvectors, see [22]. The advantage of the approach presented here is that instead of a (maybe complicated) nuclear topological vector space, a relatively simple-minded Banach space can be used.

The introduction of Gelfand triples does not only offer a better description of a self-adjoint operator but it allows also simplification of proofs. For example, in the discussion of the Fourier transform $\mathcal{F}$, the latter is considered as an object on $S_0(\mathbb{R}^d)$ where everything is well-defined, and Parseval's formula and taking the inverse Fourier transform are justified by the nice properties of $S_0(\mathbb{R}^d)$. By a density argument we get all properties of the Fourier transform on the level of $L^2(\mathbb{R}^d)$. And we obtain an extension of the Fourier transform to $S_0'(\mathbb{R}^d)$, the so-called *generalized Fourier transform*, by duality.

The preceding discussion suggests the following lemma which says that assertions for an operator on the S_0-level are actually statements for $L^2(\mathbb{R}^d)$ and S_0', respectively.

Lemma 16: *The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:*

(1) *$\mathcal{F}$ is an isomorphism from $S_0(\mathbb{R}^d)$ to $S_0(\widehat{\mathbb{R}^d})$,*
(2) *$\mathcal{F}$ is a unitary map between $L^2(\mathbb{R}^d)$ and $L^2(\widehat{\mathbb{R}^d})$,*
(3) *$\mathcal{F}$ is a weak* (as well as a norm-to-norm) continuous bijection from $S_0'(\mathbb{R}^d)$ to $S_0'(\widehat{\mathbb{R}^d})$.*

Furthermore we have that Parseval's formula

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle \tag{33}$$

is valid for $(f,g) \in S_0(\mathbb{R}^d) \times S_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(S_0, L^2, S_0')(\mathbb{R}^d)$.

The properties of Fourier transform are expressed by the *Gelfand bracket*

$$\langle f, g\rangle_{(S_0,L^2,S_0')(\mathbb{R}^d)} = \langle \hat{f}, \hat{g}\rangle_{(S_0,L^2,S_0')(\widehat{\mathbb{R}}^d)} \tag{34}$$

which combines the functional brackets of Banach spaces and of the inner-product for the Hilbert space.

The Fourier transform is a prototype for the notion of a Gelfand triple isomorphism.

Definition 17: If $(B_1, \mathcal{H}_1, B_1')$ and $(B_2, \mathcal{H}_2, B_2')$ are Gelfand triples then an operator A is called a *[unitary] Gelfand triple isomorphism* if

(1) A is an isomorphism between B_1 and B_2.
(2) A is a [unitary operator resp.] isomorphism from $\mathcal{H}_1$ to $\mathcal{H}_2$.

(3) A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between B_1' and B_2'.

In this terminology the Fourier transform is a unitary Gelfand triple isomorphism (actually an automorphism) on the Gelfand triple $(S_0, L^2, S_0')(\mathbb{R}^d)$. In the following lemma we give conditions for the extension of a linear mapping given on $S_0(\mathbb{R}^d)$ to a unitary mapping on $L^2(\mathbb{R}^d)$.

Lemma 18: *(cf. [18]) Let U be a unitary mapping from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. The mapping U extends to a Gelfand triple isomorphism between $(S_0, L^2, S_0')(\mathbb{R}^d)$ and $(S_0, L^2, S_0')(\mathbb{R}^d)$ if and only if the restriction of U to $S_0(\mathbb{R}^d)$ defines a bounded bijective linear mapping from $S_0(\mathbb{R}^d)$ onto itself.*

Due to this lemma we only have to check the properties of U at the S_0-level, i.e., to verify the existence of some $C > 0$ such that

$$\|Uf\|_{S_0(\mathbb{R}^d)} \leq C\|f\|_{S_0(\mathbb{R}^d)}. \tag{35}$$

The discussion of the Fourier transform $\mathcal{F}$ on the Gelfand triple $(S_0, L^2, S_0')(\mathbb{R}^d)$ allows to think of $\mathcal{F}$ as a bounded operator between $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$ with a distributional kernel $k(t,\omega) = e^{-2\pi i t\omega}$. The existence of a distributional kernel for any bounded operator between $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$ is *kernel theorem* for $S_0(\mathbb{R}^d)$ (cf. [15], Thm. 7.4.2). Before we give a precise description of this important fact, we recall the notion of a Wilson basis. With the help of a Wilson basis we can adapt a linear algebra reasoning to the infinite-dimensional setting.

In 1991 Daubechies, Jaffard, and Journé [8] followed an idea of Wilson in their construction of an orthonormal basis from a Gabor system $\mathcal{G}(g, \Lambda)$ of $L^2(\mathbb{R}^d)$. Wilson suggested that the building blocks $\pi(x,\omega)g$ of an orthonormal basis of $L^2(\mathbb{R}^d)$ should be symmetric in ω and should be concentrated at ω and $-\omega$.

Definition 19: For $g \in L^2$ the associated Wilson system $\mathcal{W}(g)$ consists of the functions

$$\psi_{k,n} = c_n T_{\frac{k}{2}}(M_n + (-1)^{k+n} M_{-n})g, \quad (k,n) \in \mathbb{Z} \times \mathbb{N}_0,$$

where $c_0 = \frac{1}{2}$ and $c_n = \frac{1}{\sqrt{2}}$ for $n \geq 1$, $\psi_{k,0} = T_k g$ and $\psi_{2k+1,0} = 0$ for $k \in \mathbb{Z}$.

They proved the following theorem which shows a method for the construction of a Wilson basis from a Gabor system $\mathcal{G}(g, \frac{1}{2}\mathbb{Z} \times \mathbb{Z})$. Later Feichtinger, Gröchenig, and Walnut [13] showed that Wilson systems provide an unconditional basis for $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$ endowed with the w^*-topology (actually, for all modulation spaces $M_m^{p,q}(\mathbb{R}^d)$ with $p, q < \infty$). Therefore Wilson systems provide us with a natural class of bases for time-frequency analysis. The existence of an unconditional basis for $S_0(\mathbb{R}^d)$ will be very helpful in our discussion of the kernel theorem for $S_0(\mathbb{R}^d)$ and its construction relies heavily on the functorial properties of S_0, cf. [15].

Theorem 20: Let $\mathcal{G}(g, \frac{1}{2}\mathbb{Z} \times \mathbb{Z})$ be a tight frame for $L^2(\mathbb{R})$ with $\|g\| = 1$ and $g(x) = \overline{g(-x)}$. Then the Wilson system $\mathcal{W}(g)$ is an orthonormal basis of $L^2(\mathbb{R})$.

As a corollary we get Wilson bases for $L^2(\mathbb{R}^d)$ by taking tensor products.

Corollary 21: *Let $\mathcal{W}(g)$ be a Wilson basis for $L^2(\mathbb{R})$ and define $\mathbf{\Psi}_{k,n} = \prod_{j=1}^d \psi_{r_j,s_j}$ for $(r,s) \in \mathbb{Z}^d \times \mathbb{N}_0^d$. Then $\mathbf{\Psi}_{k,n}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$.*

In applications of mathematics one often has to deal with linear systems. In the discrete and finite case each linear system is a linear mapping from the input space $\mathbb{R}^n$ into the output space $\mathbb{R}^m$ of our system and its action is given by matrix multiplication after a choice of bases in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively (similarly from $\mathbb{C}^n$ to $\mathbb{C}^m$ using complex matrices).

A linear system in infinite dimensions may be considered as a continuous analog of matrix multiplication (replacing summation by integration), i.e.,

$$g(x) = Kf(x) = \int_{\mathbb{R}^d} k(x,y) f(y) dy.$$

We can think of the input values $f(y)$ as being listed in an infinite column vector and $k(x,y)$ as an infinite matrix, the so-called *kernel* of K, and the integral $\int_{\mathbb{R}^d} k(x,y)f(y)dy$ providing the entries of the output vector in the expected way. In signal processing, such a model is known as a *linear time-variant system.*

For a wide range of function spaces (covering practically all cases relevant for applications) and by means of the use of generalized functions, this analogy can be given a precise mathematical meaning. The natural way of describing this context is via so-called *kernel theorems*. Although only Hilbert Schmidt operators can be described as integral operators with L^2-kernels, every bounded linear system A on $L^2(\mathbb{R}^d)$ can be uniquely described by some distributional kernel $K \in S_0'(\mathbb{R}^{2d})$.

The notion of Gelfand triples suggests to consider bounded linear operators between arbitrary L^p and L^q-spaces (with $p < \infty$) as bounded operators from $S_0(\mathbb{R}^d)$ to $S_0'(\mathbb{R}^d)$ (by trivial restriction of the range). Since Kf is only a distribution, we can describe it only indirectly by applying the output distribution to some test function $g \in S_0(\mathbb{R}^d)$.

Suppose we have an integral operator K with distributional kernel k on $S_0(\mathbb{R}^d)$, i.e., we think of K in a weak sense

$$\langle Kf, g\rangle = \langle k, g \otimes f\rangle, \quad f, g \in S_0(\mathbb{R}^d),$$

where $(g \otimes f)(x, y)$ denotes the tensor product $g(x)f(y)$, then K is a bounded operator between $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$. Since by duality we deduce that

$$|\langle Kf, g\rangle| = |\langle k, f \otimes g\rangle| \leq \|k\|_{S_0'} \|f \otimes g\|_{S_0} = \|k\|_{S_0'} \|f\|_{S_0} \|g\|_{S_0}$$

is true for all $g \in S_0(\mathbb{R}^d)$, we have that $Kf \in S_0'(\mathbb{R}^d)$. Therefore the operator K is bounded between $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$ with the following estimate for the operator norm of K:

$$\|K\|_{op} \leq \|k\|_{S_0'} .$$

The non-trivial aspect of the kernel theorem is that the converse is true.

Theorem 22: If K is a bounded operator from $S_0(\mathbb{R}^d)$ to $S_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in S_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in S_0(\mathbb{R}^d)$.

We only sketch a proof and refer the interested reader to the book of Gröchenig [25] for the technical details.

We define the infinite matrix $\mathbf{a} = \left(a_{(l,m),(r,s)}\right)$ of the operator K with respect to a multivariate Wilson basis $\mathcal{W}(g)$ by

$$a_{(l,m),(r,s)} = \langle K\mathbf{\Psi}_{r,s}, \mathbf{\Psi}_{l,m}\rangle. \tag{36}$$

Then the matrix $\mathbf{a}$ is bounded from $\ell^1(\mathbb{Z}^d \times \mathbb{N}_0^d)$ to $\ell^\infty(\mathbb{Z}^d \times \mathbb{N}_0^d)$. Therefore, we can define a kernel k for K as in linear algebra, by

$$k = \sum_{l,m,r,s} a_{(l,m),(r,s)} \mathbf{\Psi}_{l,m} \otimes \mathbf{\Psi}_{r,s}. \tag{37}$$

Now, we know that $\{\mathbf{\Psi}_{l,m} \otimes \mathbf{\Psi}_{r,s}\}$ is an orthonormal basis for $L^2(\mathbb{R}^{2d})$ which yields that $k \in S_0'(\mathbb{R}^{2d})$ with weak*-convergence of the sum.

An important corollary of the preceding discussion is the following observation.

Corollary 23: *Let $(\mathbf{\Psi}_{\mathbf{k},\mathbf{n}})$ be an orthonormal Wilson basis for $L^2(\mathbb{R}^d)$. Then the coefficient mapping $D : f \mapsto \langle f, \mathbf{\Psi}_{k,n}\rangle$ induces a Gelfand triple isomorphism between $(S_0, L^2, S_0')(\mathbb{R}^d)$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z}^d \times \mathbb{N}^d)$.*

Proof: Since $(\mathbf{\Psi}_{k,n})$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ the analysis operator $f \mapsto \langle f, \mathbf{\Psi}_{k,n}\rangle$ is an isomorphism between $L^2(\mathbb{R}^d)$ and $\ell^2(\mathbb{Z}^d \times \mathbb{N}^d)$. The Wilson system $(\mathbf{\Psi}_{k,n})$ is an unconditional basis for $S_0(\mathbb{R}^d)$ and therefore the analysis operator gives an isomorphism between $S_0(\mathbb{R}^d)$ and $\ell^1(\mathbb{Z}^d \times \mathbb{N}^d)$. By duality we obtain that $S_0'(\mathbb{R}^d)$ is isomorphic to $\ell^\infty(\mathbb{Z}^d \times \mathbb{N}^d)$. □

8. The spreading function

The notion of a Gelfand triple has turned out to be a very fruitful concept for investigations in Gabor analysis, see [15], [7], [10]. In this section we present some results of Feichtinger and Kozek on Gelfand triples for time-frequency analysis. All these results have their origin in the search of a mathematical framework for problems in signal analysis. Many problems in applications are modelled as linear time-variant systems (LTV). In the last section we learned that a LTV is just an integral operator K acting on signals with finite energy,

$$Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy, \qquad f \in L^2(\mathbb{R}^d). \tag{38}$$

The quality of an integral operator K on $L^2(\mathbb{R}^d)$ relies on properties of its kernel k. For example, integrability conditions on k yield classes of nice operators. The most prominent class of operators, the *Hilbert-Schmidt* operators $\mathcal{HS}$, are defined in terms of integrability conditions. Namely, an integral operator K on $L^2(\mathbb{R}^d)$ is a *Hilbert-Schmidt* operator if $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. From the analogy between integral operators and matrices we see that Hilbert-Schmidt operators are a generalization of the space of linear mappings on a finite dimensional vector space V with respect to the Frobenius norm, see (10) in Section 2.

The class of Hilbert-Schmidt operators $\mathcal{HS}$ has a natural inner product. Let $K_1, K_2 \in \mathcal{HS}$ with kernels k_1, k_2, respectively. Then

$$\langle K_1, K_2\rangle_{\mathcal{HS}} := \langle k_1, k_2\rangle_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \tag{39}$$

defines an inner product on $\mathcal{HS}$. The associated *Hilbert-Schmidt norm* $\|K\|_{\mathcal{HS}} := \big(\langle K, K\rangle_{\mathcal{HS}}\big)^{1/2}$ gives $\mathcal{HS}$ the structure of a Hilbert space [39]. Furthermore we recall that every Hilbert-Schmidt operator in $\mathcal{HS}$ is a compact operator on $L^2(\mathbb{R}^d)$. Recall that a compact operator K on $L^2(\mathbb{R}^d)$ is of Hilbert-Schmidt type if and only if there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$ and a sequence of scalars $(\lambda_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ such that

$$Kf = \sum_{n\in\mathbb{N}} \lambda_n \langle e_n, f\rangle e_n. \tag{40}$$

The sequence of scalars $(\lambda_n)_{n\in\mathbb{N}}$ are actually the eigenvalues of K and $\|K\|_{\mathcal{HS}} = \big(\sum_{n\in\mathbb{N}} |\lambda_n|^2\big)^{1/2}$. The space of Hilbert-Schmidt operators $\mathcal{HS}$ is not closed in the C^*-algebra $\mathcal{K}$ of compact operators on $L^2(\mathbb{R}^d)$ with respect to the operator norm, and there exist compact operators which are not of Hilbert-Schmidt type. But $\mathcal{HS}$ is a two-sided ideal in $\mathcal{K}$.

If we choose as orthonormal basis of $L^2(\mathbb{R}^d)$ a Wilson basis $(\mathbf{\Psi}_{\mathbf{k},\mathbf{n}})$, then the preceding observations lead to an isomorphism between $\mathcal{HS}$ and $\ell^2(\mathbb{Z}^d \times \mathbb{N}^d)$. Now we can make use of the concept of Gelfand triples, but this time we take the Hilbert-Schmidt operators as Hilbert space of an *Operator Gelfand triple*. We observe that the kernel theorem for $S_0(\mathbb{R}^d)$ provides us with another class of operators with "smooth kernels". We write $\mathcal{L}$ for the space of bounded linear operators on a Banach space B. One finds that $K \in \mathcal{L}(S_0'(\mathbb{R}^d), S_0(\mathbb{R}^d))$ can be identified with kernels $k \in S_0'(\mathbb{R}^{2d})$ which is dense in $\mathcal{HS}$. But the class of Hilbert-Schmidt operators $\mathcal{HS}$ is dense in $\mathcal{L}(S_0(\mathbb{R}^d), S_0'(\mathbb{R}^d))$ and therefore $(\mathcal{L}(S_0'(\mathbb{R}^d), S_0(\mathbb{R}^d)), \mathcal{HS}, \mathcal{L}(S_0(\mathbb{R}^d), S_0'(\mathbb{R}^d)))$ is indeed a Gelfand triple. In this setting the kernel theorem can be interpreted as a unitary Gelfand triple isomorphism between this triple of operators and their kernels in $(S_0, L^2, S_0')(\mathbb{R}^d\times\mathbb{R}^d)$. There is another Gelfand triple isomorphism that associates the $\mathcal{HS}$ Gelfand triple with the Gelfand triple $(S_0, L^2, S_0')(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)$: the so-called *spreading symbol* of operators.

As a motivation we discuss a problem of great practical interest: communication with cellular phones. In modern communication cellular phones play a crucial rule in everyday life. How do engineers solve the problem of transmitting a signal f from a sender A to a receiver B? In the most general situation, sender A and receiver B move in different directions with certain velocities, which leads to a variation of the path lengths of the transmitted signal f and, due to the Doppler effect, to a change of frequencies. Therefore B receives a signal of the following form

$$\tilde{f} = \iint_{\mathbb{R}^2} \eta(K)(x,\omega) M_\omega T_x f dx d\omega, \tag{41}$$

where the function $\eta(K)$ models the effect of the channel by the amount of time-frequency shifts arising as just described, applied to the signal f.

The receiver B is not interested in the signal $\tilde{f}$ but in the original signal f. From a mathematical point of view, $\tilde{f}$ is just the action of an operator K on the signal f, i.e., $\tilde{f} = Kf$. In this picture B has to invert the operator K to get the information contained in the signal f. Operators of this form are called *pseudo-differential operators* and arise naturally in many problems of physics, engineering and mathematics. The function $\eta(K)$ is the so-called *spreading function* of the operator K. In the following, we look for conditions on the spreading function $\eta(K)$ which allow an inversion of our pseudo-differential operator K.

First the equation (41) suggests a decomposition of a general operator K on $L^2(\mathbb{R}^d)$ as a continuous superposition of time-frequency shifts.

$$K = \iint_{\mathbb{R}^{2d}} \eta(K)(x,\omega) M_\omega T_x dx d\omega. \tag{42}$$

We already know such a decomposition of the identity operator on $L^2(\mathbb{R}^d)$ since this is the inversion formula for the STFT:

$$I_{L^2(\mathbb{R}^d)} = \frac{1}{\langle g,\gamma\rangle} \iint_{\mathbb{R}^{2d}} V_g f(x,\omega) M_\omega T_x dx d\omega \tag{43}$$

for $g,\gamma \in L^2(\mathbb{R}^d)$ with $\langle g,\gamma\rangle \neq 0$.

The non-commutativity of translation and modulation operators on $L^2(\mathbb{R}^d)$ leads to a twisted convolution of the spreading functions of two operators K and L. Let $K, L \in \mathcal{L}(S_0, S_0')$ and $\eta(K), \eta(L)$ their spreading functions. Then the spreading function of the composition KL is given by *twisted convolution* of $\eta(K)$ and $\eta(L)$:

$$\eta(KL)(x,\omega) = \iint_{\mathbb{R}^2} \eta(K)(x',\omega')\eta(L)(x-x',\omega-\omega')e^{-2\pi i x'(\omega-\omega')}d\omega'. \quad (44)$$

The spreading function of the adjoint operator K^* is given by

$$\eta(K^*)(x,\omega) = \overline{\eta(K)(-x,-\omega)} \cdot e^{-2\pi i x\omega} \quad (45)$$

and therefore leads to a noncommutative involution. Later we will return to this topic in the context of Gröchenig/Leinert's resolution of the "irrational case"-conjecture [26].

The relation between the kernel k of an operator K from the Gelfand triple $(\mathcal{L}(S_0, S_0'), \mathcal{HS}, \mathcal{L}(S_0', S_0))$ and its spreading function $\eta(K)$ is given by

$$\eta(K)(x,\omega) = \int_{\mathbb{R}^d} k(y, y-x)e^{-2\pi i y\omega}dy, \quad (46)$$

which is very useful in the calculation of the spreading function of an operator K. It can be interpreted literally at the lowest level (integrals etc. exist), and extends by continuity to the "upper levels". Moreover, it can be described by the fact that it is the unique Gelfand triple isomorphism which maps TF-shift operators onto the corresponding Dirac measures in the TF-plane (hence reproducing exactly the situation we had in the finite case).

The spreading function of an operator K is an object living on the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Therefore a further understanding of its properties is necessary according to the structure of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ which is closely related to the structure of the Euclidean plane $\mathbb{R}^d \times \mathbb{R}^d$. Namely, the time-frequency plane is a symplectic manifold, i.e., there exists a non-degenerate 2-form $\Omega(X,Y) = y \cdot \omega - x \cdot \eta$ for two points $X = (x,\omega), Y = (y,\eta)$ in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Since Ω is non-degenerate there is a unique invertible skew-symmetric linear operator $\mathcal{J}$ on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ such that the symplectic form Ω and the Euclidian inner product are related as follows: $\Omega(X,Y) = \langle \mathcal{J}X, Y\rangle$ for all $X, Y \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. This implies an important fact about the characters of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Namely, the characters are given by $\{\chi_s(X,Y) = e^{2\pi i\Omega(X,Y)} | X \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}$ for a fixed $Y \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Therefore it is natural to analyse a function F on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ with the *symplectic Fourier Transform*

$$\mathcal{F}_s F(X) = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(Y)e^{2\pi i\Omega(X,Y)}dY \quad (47)$$

instead of the Fourier transform $\mathcal{F}$ induced by the standard inner-product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d \times \mathbb{R}^d$. From the relation between symplectic form and inner-product we obtain that the symplectic Fourier transform $\mathcal{F}_s$ is just a Fourier transform followed by a rotation by $\frac{\pi}{2}$ since $\mathcal{J}$ describes a rotation by $\frac{\pi}{2}$ around the origin of $\mathbb{R}^d \times \mathbb{R}^d$. This fact allows us to derive similiar statements for the symplectic Fourier transform as for the Euclidian Fourier transform.

(1) $\mathcal{F}_s$ is a unitary mapping from $L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ onto $L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.
(2) $\mathcal{F}_s^{-1} = \mathcal{F}_s$ (involutive property);
(3) $\mathcal{F}_s \left(S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d) \right) = S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

By duality we obtain that

Proposition 24: *The symplectic Fourier transform $\mathcal{F}_s$ defines a unitary Gelfand triple automorphism on $(S_0, L^2, S_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.*

Another reason for our choice of $S_0(\mathbb{R}^{2d})$ as space of test functions is that the Poisson summation formula for symplectic Fourier transform holds pointwise and with absolute convergence. Recently, we have shown that the Fundamental Identity of Gabor Analysis can be dervied by an application of Poisson summation to a product of two STFT's:

Theorem 25: Let Λ a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ with adjoint lattice Λ° and $F \in S_0(\mathbb{R}^{2d})$. Then

$$\sum_{\lambda \in \Lambda} F(\lambda) = \frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} \mathcal{F}_s F(\lambda^\circ) \tag{48}$$

holds pointwise and with absolute convergence on both sides.

The spreading function is an important tool for the description of (slowly) time-variant channels in communication theory, but it is not the only symbol associated with a linear operator. In the theory of pseudo-differential operators the *Kohn-Nirenberg symbol* (KN), denoted by $\sigma(K)$, is used for an operator $K \in (S_0, L^2, S_0')(\mathbb{R}^d)$. It is defined as the symplectic Fourier transform of the spreading function $\eta(K)$:

$$\sigma(x, \omega) = \mathcal{F}_s \eta(K) = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \eta(K) e^{2\pi i (y \cdot \omega - x \cdot \eta)} dy d\eta, \quad (x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d. \tag{49}$$

If $Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$ then $\sigma(K) = \int_{\mathbb{R}^d} k(x,x-y)e^{-2\pi i y\cdot\omega}dy$. In signal analysis $\sigma(K)$ was introduced by Zadeh and is called the *time-varying transfer function* of a system modelled by K. As an example we mention the KN symbol of a rank-one operator $f \otimes \overline{g}$, which describes the mapping $h \mapsto \langle h, g\rangle f$, is equal to

$$\sigma(f \otimes \overline{g})(x,\omega) = f(x)\overline{\hat{g}(\omega)}e^{-2\pi i x\cdot\omega}, \qquad (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d, \tag{50}$$

the Rihaczek distribution of f against g. For $f, g \in S_0(\mathbb{R}^d)$ we have that the KN-symbol $\sigma(f \otimes \overline{g})$ is in $S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ which in turn implies (using the last equation) that $(x,\omega) \mapsto e^{2\pi i x\cdot\omega}$ is a pointwise multiplier on $S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

After these preparations we can state one of our main results:

Theorem 26: The spreading function $K \mapsto \eta(K)$ is a unitary Gelfand triple isomorphism from $(\mathcal{L}(S_0, S_0'), \mathcal{HS}, \mathcal{L}(S_0', S_0))$ to $(S_0, L^2, S_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

Corollary 27: *The KN symbol of K induces a unitary Gelfand triple isomorphism between $(\mathcal{L}(S_0, S_0'), \mathcal{HS}, \mathcal{L}(S_0', S_0))$ and $(S_0, L^2, S_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.*

Another consequence of the preceding theorem are the following Gelfand-bracket identities for $K_1, K_2 \in (\mathcal{L}(S_0, S_0'), \mathcal{HS}, \mathcal{L}(S_0', S_0))$:

$$\langle K_1, K_2\rangle_{(\mathcal{B},\mathcal{HS},\mathcal{B}')} = \langle \eta(k_1), \eta(k_2)\rangle_{(S_0,L^2,S_0')(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \tag{51}$$

$$= \langle \sigma(k_1), \sigma(k_2)\rangle_{(S_0,L^2,S_0')(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}, \tag{52}$$

with $\mathcal{B} = \mathcal{L}(S_0, S_0')$ and $\mathcal{B}' = \mathcal{L}(S_0', S_0)$ respectively.

The KN symbol of a rank-one operator $f \otimes \overline{g}$, is the Rihaczek distribution and by an application of the (inverse) symplectic Fourier transform we get another time-frequency distribution: the STFT!

Lemma 28: *For $f, g \in S_0(\mathbb{R}^d)$ the rank-one operator $f \otimes \overline{g}$ has a kernel in $S_0(\mathbb{R}^d)$. Moreover the corresponding spreading function is*

$$\eta(f \otimes \overline{g})(x,\omega) = \int_{\mathbb{R}^d} f(x)\overline{g(y-x)}e^{-2\pi i y\cdot\omega}dy \tag{53}$$

and hence coincides with $V_g f \in S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

In the light of this result the inversion formula for the STFT is a superposition of time-frequency shifts with the spreading function of the rank-one operator $g \otimes \overline{\gamma}$ for $g, \gamma \in L^2(\mathbb{R}^d)$ with $\langle g, \gamma \rangle \neq 0$:

$$f = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \eta(f \otimes \overline{g})(x, \omega) T_x M_\omega \gamma \, dx d\omega. \tag{54}$$

Recall that in analogy with the characters $\{\chi_\omega : \omega \in \widehat{\mathbb{R}^d}\}$, the time-frequency shifts $\{\pi(X) : X = (x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}$ would be an orthonormal set with respect to the Hilbert-Schmidt inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}S}$ and $\eta(f \otimes \overline{g})(x, \omega) = \langle f \otimes \overline{g}, \pi(x, \omega) \rangle_{\mathcal{H}S}$ but as in the case of Fourier transform, the building blocks $\pi(X)$ for $X \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ of our orthonormal system $\{\pi(X) : X = (x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}$ are not Hilbert- Schmidt. As in our treatment of the Fourier transform, it is not so important that the building blocks are elements of our Hilbert space but that they allow us to get expressions as it were an orthonormal set of elements in our Hilbert space.

As a first example we state a generalization of the inversion formula for the STFT from $L^2(\mathbb{R}^d)$ to the Gelfand triple $(S_0, L^2, S_0')(\mathbb{R}^d)$, where for $f \in S_0'(\mathbb{R}^d)$ the formula is interpreted in a weak sense.

Proposition 29: *Let $g, \gamma \in S_0(\mathbb{R}^d)$ with $\langle g, \gamma \rangle \neq 0$. Then*

$$f = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \eta(f \otimes \overline{g})(x, \omega) T_x M_\omega \gamma \; dx d\omega. \tag{55}$$

holds for $f \in (S_0, L^2, S_0')(\mathbb{R}^d)$.

That is a special case of a general statement about the spreading function.

Theorem 30: Any $K \in (\mathcal{L}(S_0, S_0'), \mathcal{H}S, \mathcal{L}(S_0', S_0))$ has a representation

$$K = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle K, \pi(x, \omega) \rangle_{\mathcal{L}(S_0, S_0')} \, \pi(x, \omega) \, dx d\omega \tag{56}$$

convergent in the strong resp. weak*-sense. The (complex-valued) amplitude function arising in this context, i.e. $\eta(K)(x, \omega) = \langle K, \pi(x, \omega) \rangle_{\mathcal{L}(S_0, S_0')}$, is called the spreading distribution of the operator K.

The basic tool in the proof is the fact that the spreading representation maps a time-frequency shift $\pi(X)$, for $X = (x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, on the Dirac measure δ_X, i.e., $\eta(\pi(X)) = \delta_X$, and the relation between the spreading function and the kernel of an operator K.

The preceding theorem is the mathematical justification of a widely used statement that the spreading function of an operator K is a measure for the time-frequency content of K.

In our intuition we move an operator K over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and expect a simple relation between the original symbol of K and the symbol after a movement to $(x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. The KN-symbol of an operator K is shifted by $T_{x,\omega}$ in the time-frequency plane.

Lemma 31: *Let K belong to one of the spaces $(\mathcal{L}(S_0', S_0), \mathcal{HS}, \mathcal{L}(S_0', S_0))$, then $\pi(x,\omega)K\pi(x,\omega)^*$, the conjugation of K by $\pi(x,\omega), (x, omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, corresponds to translation of the KN symbol $\sigma(K)$,*

$$\sigma(\pi(x,\omega)K\pi(x,\omega)^*) = T_{(x,\omega)}\big(\sigma(K)\big). \tag{57}$$

This property of the KN symbol is of central importance in our study of the Gabor frame operator to which we devote the final part of this section. Let $\mathcal{G} = (g, \Lambda)$ be a Gabor system for a lattice $\Lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then the Gabor frame operator $S_{g,\Lambda}$ commutes with all time-frequency shifts of the lattice Λ, i.e.

$$\pi(\lambda) S_{g,\Lambda} \pi(\lambda)^* = S_{g,\Lambda}, \quad \text{for all} \quad \lambda \in \Lambda. \tag{58}$$

This fact was the motivation for Feichtinger and Kozek to introduce the class of Λ-invariant operators [15].

Definition 32: Let Λ be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and K an operator in $\mathcal{B}(\Lambda)$. Then K is called Λ-*invariant* if $\pi(\lambda)K = K\pi(\lambda)$ for all $\lambda \in \Lambda$.

In the following we want to find the support of the spreading function $\eta(K)$ of an Λ-invariant operator $K \in (\mathcal{L}(S_0, S_0'), \mathcal{HS}, \mathcal{L}(S_0', S_0))$. As a first step towards this result we study spreading representations of K on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

Lemma 33: *Let $K_1, K_2 \in \mathcal{L}(S_0, S_0')$ with spreading function $\eta(K_1), \eta(K_2)$, respectively. Then*

(1) $\eta(K_1K_2)(\lambda) = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \eta(K_1)(\mu)\eta(K_2)(\lambda - \mu)\rho(\lambda - \mu, \mu)d\mu$ *with* $\rho(X,Y) = e^{2\pi i(y\cdot\omega - x\cdot\eta)}$ *for* $X = (x,\omega), Y = (y,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.
(2) $supp(\eta(K_1)\eta(K_2)) \subset supp(K_1) + supp(K_2)$.
(3) $|\eta(K_1K_2)| = |\eta(K_1)| * |\eta(K_2)|$ *for* $\eta(K_1), \eta(K_2) \in L^1_{loc}(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

The proof of (i) is a consequence of the commutation relation for time-frequency shifts and the fact that for $K_1 \in \mathcal{L}(S_0, S_0')$ and $K_2 \in \mathcal{L}(S_0, S_0')$ also $K_1 K_2 \in \mathcal{L}(S_0', S_0)$. Now each operator K in $\mathcal{L}(S_0, S_0')$ has an absolutely convergent spreading representation and therefore our result holds pointwise. The support condition follows from the analogous result for the ordinary convolution.

By abstract reasons, each Λ-invariant operator K has a representation in the set of all operators concentrated on $\Lambda^\circ = \{\lambda^\circ \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d | \pi(\lambda)\pi(\lambda^\circ) = \pi(\lambda^\circ)\pi(\lambda)\}$ since K lies in the commutant of the (C^*, von Neumann) algebra generated by $\{\pi(\lambda) : \lambda \in \Lambda\}$. The set Λ° is the so-called *adjoint lattice*, since it is the annihilator subgroup of Λ for the symplectic Fourier transform $\mathcal{F}_s$, and if $\Lambda^\perp$ is the annihilator subgroup of Λ with respect to $\mathcal{F}$, then $\Lambda^\circ = \mathcal{J}\Lambda^\perp$.

The time-frequency invariance of $S_0(\mathbb{R}^d)$ implies that K and $\pi(\lambda)K$ are in the Gelfand triple $(\mathcal{L}(S_0, S_0'), \mathcal{HS}, \mathcal{L}(S_0', S_0))$, too. Therefore, the Λ-invariance of T translates into a periodicity condition for the symbol $\sigma(K)$

$$\sigma(K) = T_\lambda(\sigma(K)), \quad \lambda \in \Lambda\,. \tag{59}$$

This periodicity condition corresponds to a support condition for the spreading function since But $\{e^{-2\pi\Omega(\lambda,\mu)} | \lambda \in \Lambda\}$ for a fixed $\mu \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is a group of characters on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ yields that

$$\mathrm{supp}(\eta(K)) \subset \mathcal{J}\Lambda^\perp = \Lambda^\circ. \tag{60}$$

The fact that distributions in $S_0'(\mathbb{R}^d)$ with support in a discrete subgroup is a sum of Dirac measures with a bounded sequence of coefficients implies, that for some bounded sequence (c_{λ°) over Λ°

$$\eta(K) = \sum_{\lambda^\circ \in \Lambda^\circ} c_{\lambda^\circ} \delta_{\lambda^\circ} \tag{61}$$

with $c_{\lambda^\circ} = (K)_{\lambda^\circ} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d / \Lambda^\circ} \sigma(K)(\mu) e^{2\pi i \Omega(\lambda,\mu)} d\mu$.

Returning to the description in the operator domain we arrive at the following characterization

Theorem 34: Let $K \in (\mathcal{L}(S_0, S_0'), \mathcal{HS}, \mathcal{L}(S_0', S_0))$ and $\sigma(K)$ the KN symbol. Then $\sigma(K)$ is a Λ-periodic distribution with a symplectic Fourier transform supported on Λ°. Furthermore

$$K = \sum_{\lambda^\circ \in \Lambda^\circ} (K)_{\lambda^\circ} \pi(\lambda^\circ). \tag{62}$$

Corollary 35: *The mapping* $\sigma(K) \mapsto (K)_{\lambda^\circ}$ *is a unitary Gelfand triple isomorphism between* $(S_0, L^2, S_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d/\Lambda)$ *and* $(\ell^1, \ell^2, \ell^\infty)(\Lambda^\circ)$.

Note that the time-frequency invariance of $S_0(\mathbb{R}^d)$ implies the boundedness of K on $S_0(\mathbb{R}^d)$ since

$$\|K\|_{\mathcal{L}(S_0)} = \| \sum_{\lambda^\circ \in \Lambda^\circ} (K)_{\lambda^\circ} \pi(\lambda^\circ) \|_{\mathcal{L}(S_0)} \leq \sum_{\lambda^\circ \in \Lambda^\circ} |(K)_{\lambda^\circ}|. \tag{63}$$

The next theorem shows that for any Λ-invariant operator K with $\sigma(K) \in S_0'((\mathbb{R}^d \times \widehat{\mathbb{R}}^d)/\Lambda)$ there exists a prototype operator $P \in \mathcal{L}(S_0, S_0')$ such that periodization of P in the time-frequency plane corresponds to sampling of the spreading function $\eta(P)$ on Λ°.

Theorem 36: Let K be a Λ-invariant operator with $\sigma(K) \in S_0'((\mathbb{R}^d \times \widehat{\mathbb{R}}^d)/\Lambda)$. Then there exists some $P \in \mathcal{L}(S_0, S_0')$ such that its periodization is exactly K

$$K = \sum_{\lambda \in \Lambda} \pi(\lambda)\, P\, \pi(\lambda)^* = \frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} \langle P, \pi(\lambda^\circ) \rangle_{\mathcal{L}(S_0, S_0')}\ \pi(\lambda^\circ). \tag{64}$$

Remark 37: The preceding result is a discrete analog of our spreading representation for operators in $\mathcal{L}(S_0, S_0')$ which, in the context of Gabor analysis, leads to the so-called *Janssen representation* of the Gabor frame operator.

The proof of the theorem is based on two important features of the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

(1) $\{U \mapsto \pi(\lambda)\, U\, \pi(\lambda)^* | \lambda \in \Lambda\}$ defines a unitary representation of Λ which gives the Λ-invariance of K.
(2) An application of the Poisson summation formula for the symplectic Fourier transform to $\sigma(P)$ with respect to the lattice Λ maps the periodization of

$$\sigma(K) = \sum_{\lambda \in \Lambda} T_\lambda(\sigma(P)) \tag{65}$$

to the sampling of the spreading function $\eta(P)$ on the lattice Λ°.

As an application we state that the Gabor frame operator $S_{g,\Lambda}$ of a Gabor system $\mathcal{G}(g, \Lambda)$ with $g \in S_0(\mathbb{R}^d)$ is generated by shifting a rank-one operator along the lattice Λ. In addition, we use the fact that the spreading function of a rank-one operator is the STFT. Altogether we therefore have

$$S_{g,\Lambda} = \frac{1}{|\Lambda|} \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, \pi(\lambda^\circ)\gamma\rangle \pi(\lambda^\circ) \qquad (66)$$

with $\gamma \in S_0(\mathbb{R}^d)$. The last equation (66) is the so-called Jannsen representation of $S_{g,\Lambda}$ which decomposes $S_{g,\Lambda}$ into an *absolutely convergent* series of time-frequency shifts. In (66) we used implicitely another pleasant property of $S_0(\mathbb{R}^d)$.

Lemma 38: *Let $g, \gamma \in S_0(\mathbb{R}^d)$ and Λ a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then (g, γ) satisfies Tolimieri-Orr's condition (A'):*

$$\sum_{\lambda \in \Lambda} |\langle g, \gamma_\lambda\rangle| < \infty, \qquad (A').$$

This stability of Condition (A') for $g, \gamma \in S_0(\mathbb{R}^d)$ with respect to a variation of the lattice makes Feichtinger's algebra such an important object in Gabor analysis. In a recent work Feichtinger and Kaiblinger have drawn some deep consequences from this fact. Roughly speaking, they proved that the set of functions in $S_0(\mathbb{R}^d)$ which generate a Gabor frame is *"open"* [14].

We close our discussion of the Gabor frame operator with a striking result of Gröchenig/Leinert on the quality of the canonical dual of a Gabor system $\mathcal{G}(g, \Lambda)$ generated by a window $g \in S_0(\mathbb{R}^d)$.

Theorem 39: Let $g \in S_0(\mathbb{R}^d)$ and $\mathcal{G}(g, \Lambda)$ a Gabor frame of $L^2(\mathbb{R}^d)$. Then $\gamma_0 = S_{g,\Lambda}^{-1} g$ is in $S_0(\mathbb{R}^d)$.

The proof is based on a noncommutative version of Wiener's lemma for the Banach algebra $\ell^1(\Lambda)$ with twisted convolution $\sharp$ as product, and noncommutative involution $*$ as described above for the spreading function of a product of two operators in $\mathcal{L}(S_0, S_0')$ and the spreading function of the adjoint of an operator in $\mathcal{L}(S_0, S_0')$. A special case of their main result is that $(\ell^1, \sharp, *)$ is a *symmetric* Banach algebra. In this context the Wiener lemma is expressed as the inverse-closedness of the Banach algebra

$$\mathcal{A}(\Lambda) = \{A \in \mathcal{B}(L^2(\mathbb{R}^d)) \,|\, A = \sum_{\lambda \in \Lambda} a_\lambda \pi(\lambda),\ (a_\lambda) \in \ell^1(\Lambda)\}$$

of absolutely convergent time-frequency series in the C^*-algebra $C^*(\Lambda)$ generated by time-frequency shifts $\{\pi(\lambda) : \lambda \in \Lambda\}$. In other words, the argument is based on the highly non-trivial fact that an element of $\mathcal{A}(\Lambda)$ which is invertible in $C^*(\Lambda)$ has its inverse already in $\mathcal{A}(\Lambda)$.

9. Conclusion and outlook

It was the goal of this report to show how practical questions in signal processing lead to a very powerful mathematical model, the so-called Gabor frames, that induce highly structured operators which, in turn, open exciting viewpoints to advanced concepts in harmonic analysis. These concepts, such as the Segal algebra $S_0(\mathbb{R}^d)$, the Gelfand triple, and the spreading function provide deep insights into general properties of such a class of operators and relate Gabor analysis to other fields of physics and mathematics.

We have tried to give a consistent overview of the main tools of time-frequency analysis that are used to study signal expansions by means of Gabor systems. At the same time we also wanted to present state-of-the-art techniques for extracting the relevant parts of the models in order to draw a more transparent picture of the topic.

The central object of this paper is the Gabor frame-type operator. On one hand, it reduces to a sparse matrix in the finite dimensional case and can be treated by standard numerical methods. On the other hand, it is a special case of a class of Λ-periodic time-frequency operators that have a very attractive description in form of symbols over suitable function and distribution spaces. Specifically, the spreading function, the KN symbol and the Gelfand triple $(S_0, L^2, S_0')(\mathbb{R}^d)$ were introduced and studied to embed those results of the Gabor frame-type operator that initiated and popularized the Gabor theory in the mid nineties [9,28] in a larger framework of time-frequency analysis. The original ideas go back to the fundamental paper of Feichtinger and Kozek in 1998 [15]. These ideas have been evolving over time and other results, many of them nicely presented in [25], contributed to a better perception.

In the final part of the last section we returned to the central object, the Gabor frame-type operator, in order to stress the close connection between Gabor analysis and advanced time-frequency topics. The attentive reader, however, has certainly realized that it was not intended to wrap up all the relevant results. Indeed, the story is far from being complete and far from being ended. As a final taste in this sense, we shortly describe two more interesting topics. The first one is about so-called Gabor multipliers. The second one deals with the question of sampling time signals and studies Gabor expansions when increasing the sampling rate, a problem which has recently been analyzed by Kaiblinger.

In [17] Feichtinger and Nowak describe the foundation of a theory of (regular) *Gabor multipliers*, which are operators obtained by going from signal domain to some transform domain, and applying a pointwise multiplication operator before resynthesis. More generally speaking,

Definition 40: Assume we have $g_1, g_2 \in L^2(\mathbb{R}^d)$, Λ a lattice of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and let $\mathbf{m} = (m(\lambda))_{\lambda\in\Lambda}$ be a complex-valued sequence on Λ. Then the *Gabor multiplier* associated to the triple (g_1, g_2, Λ) with (upper) symbol $\mathbf{m}$ is defined by

$$G_{\mathbf{m}}(f) := G_{g_1,g_2,\Lambda,\mathbf{m}}(f) = \sum_{\lambda\in\Lambda} m(\lambda)\langle f, \pi(\lambda)g_1\rangle \pi(\lambda)g_2. \qquad (67)$$

Therefore Gabor multipliers are infinite linear combinations of rank-one operators $f \mapsto \langle f, \pi(\lambda)g_1\rangle\pi(\lambda)g_2$ with coefficients given by $(m(\lambda))_{\lambda\in\Lambda}$. The function g_1 is called the *analysis window* and g_2 the *synthesis window* of the Gabor multiplier $G_{\mathbf{m}}$. A basic question arises naturally in this context. Namely, how the properties of the Gabor multiplier $G_{\mathbf{m}}$ depends on the decay of the multiplier sequence $(m(\lambda))_{\lambda\in\Lambda}$, the time-frequency concentration of g_1, g_2 and the time-frequency lattice Λ. In general g_1 and g_2 should be *Bessel atoms* with respect to the given lattice Λ and the strong symbol $\mathbf{m}$ is assumed to be bounded. In this case the coefficient mapping C_{g_1} using the analysis window g_1, mapping f to the sequence of sampling values $V_g f$ over Λ, is bounded from $L^2(\mathbb{R}^d)$ to $\ell^2(\Lambda)$, and also the synthesis mapping D_{g_2}, mapping a sequence $\mathbf{c} = (c(\lambda))_{\lambda\in\Lambda}$ to $\sum_{\lambda\in\Lambda} c(\lambda)\pi(\lambda)g_2$, is bounded from $\ell^2(\Lambda)$ to $L^2(\mathbb{R}^d)$ and thus the Gabor multiplier $G_{\mathbf{m}} = D_{g_2}C_{g_1}$ is bounded on $L^2(\mathbb{R}^d)$.

After our discussion of Gabor frame-type operators the interested reader will already conjecture that Gabor multipliers $G_{\mathbf{m}}$ with g_1 and g_2 have very nice properties. Furthermore, the terminology of Gelfand triples allows an elegant formulation of statements about Gabor multipliers. The following result is one of the main results in [17].

Theorem 41: For every pair (g_1, g_2) in $S_0(\mathbb{R}^d)$, and any lattice $\Lambda \in$

$\mathbb{R}^d \times \widehat{\mathbb{R}^d}$, the mapping from the multiplier $(m(\lambda))_{\lambda\in\Lambda}$ to the associated Gabor multiplier $G_{g_1,g_2,\mathbf{m},\Lambda}$ maps the Gelfand triple $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$ into the bounded operators with kernel in the corresponding Gelfand triple $(S_0, L^2, S_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

For more information about Gabor multipliers, e.g., their eigenvalue behavior, and their relation to time-varying filters we refer the reader to [17].

The work [30] of Kaiblinger on the approximation of the dual Gabor window γ of a Gabor frame $\mathcal{G}(g, \Lambda)$ in $L^2(\mathbb{R}^d)$ is interesting for two reasons: On one hand, it shows that our study of finite-dimensional Gabor frames is not only a motivation for the treatment of Gabor frames for $L^2(\mathbb{R}^d)$. On the other, hand all his results about approximation of a continuous function by finite methods only work for a Gabor atom g in $S_0(\mathbb{R}^d)$.

Kaiblinger's result are based on a synthesis of fast algorithms for the computation of dual window of a Gabor frame in $\mathbb{C}^n$ and on the fact that these Gabor frames for $\mathbb{C}^n$ are obtained in a simple way from the original Gabor frame $\mathcal{G}(g, \Lambda)$. He also showed that the approximate dual windows converge not just in $L^2(\mathbb{R}^d)$ but indeed in $S_0(\mathbb{R}^d)$, which implies the convergence of the corresponding frame operators in the operator norm on $L^2(\mathbb{R}^d)$. For further information we refer to [30].

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