The Hopf equation
A toy model of fluid mechanics

1. Main physical features

▸ Mathematical description of a continuous medium

At the microscopic level, a fluid is a collection of interacting particles (Van der Waal forces, electromagnetic interactions,. . .)
$\Rightarrow$ very complex description due to the large number of particles

At the macrosopic level, a fluid can be considered as a continuous medium described by observables
$\Rightarrow$ the rheology (compressibility, viscosity, thermal conductivity,. . .) is determined by experimental data

Here we focus on inviscid (or weakly viscous) fluids such as

- air surrounding planes
- water in oceans or lakes

Note that good approximations depend on the observation scale.

The fluid motion depends also on

- **the domain** (walls, obstacles, free surface,...) and the boundary conditions ;

- **the external forces** (gravity, wind forcing, Coriolis force,...) ;

- **the initial state**.

The resulting flow is therefore complex.

For instance, the oceanic motion can be decomposed as a superposition of various fluctuations with respect to the rigid rotation

- oscillations with small periods (ripples, swell)
- oscillations with large period (storm waves, tsunamis, tides)
- thermohaline circulation (surface and deep water currents)

- Finite speed of propagation

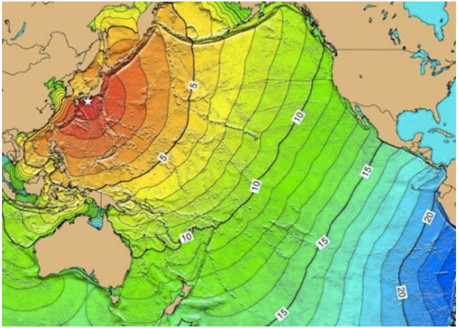

Fig. 1. Propagation of tsunamis

Contrary to what happens for dissipative processes (such as heat propagation), there is **no smoothing effect**.

Singularities (such as tsunamis) are propagated, with finite speed of propagation.

- Formation of singularities

Fig. 2. Rogue waves

Waves do not propagate independently.

Because of this **coupling**, singularities may appear without external energy source.

- Thermodynamic laws

First principle of thermodynamics : the energy is invariably conserved

For an inviscid fluid, there is no viscous dissipation.
$\Rightarrow$ the total energy is the sum of the kinetic energy, the potential energy (due to external forces) and the thermal energy (if any).

Second principle of thermodynamics : the entropy almost always increases.

In the presence of singularities, the fluid cannot be everywhere at thermodynamic equilibrium.
$\Rightarrow$ microscopic statistical fluctuations are responsible for the irreversibility of the evolution.

2. Lagrangian and Eulerian description

- The point of view of fluid particles

Fig. 3. Following the trajectories

The unknowns are the position $X_y(t)$ and velocity $V_y(t)$, for each y.
In fluid mechanics, y is a continuous parameter (related for instance to the initial position).

- The point of view of the observer

Fig. 4. Measuring the flow

At each point x of the fluid, one can measure the bulk velocity $u(t, x)$, the pressure $p(t, x)$, the temperature. . ..
Depending on the state relation of the fluid, one can then recover the statistical distribution of particles.

▸ The Hopf equation

To predict the evolution of these observables, we combine

- the fundamental principle relating forces and acceleration,
- the physical properties of the fluid (compressibility, viscosity, state relation. . .)

which leads to some system of **partial differential equations** (PDEs), relating the derivatives with respect to time ∂_t to the spatial derivatives ∂_{x_i} of the observables.

The simplest model exhibiting the physical features mentioned in the introduction is the Hopf equation

$$\begin{aligned} \partial_t u + u\partial_x u = 0, \quad t \in \mathbb{R}^+, x \in \mathbb{R}, \\ u(0,x) = u_0(x)\,. \end{aligned} \tag{1}$$

It has no real physical meaning. But the same kind of behavior is obtained for instance for the 1D compressible Euler equations (perfect gases), or the system of elasticity.

Main mathematical features

- scalar : for all t, x, $u(t, x) \in \mathbb{R}$ (without much loss of generality)
- one-dimensional : $x \in \mathbb{R}$ (channel with translation invariance)
  no easy extension in multi-D
- **hyperbolic** : finite speed of propagation
- **nonlinear** : formation of singularities (even for smooth initial data)

In the case of the Hopf equation, one can compute explicitly the solutions, and will have a very precise description of singularities.

This is not the case for general hyperbolic system of conservation laws, even not for scalar hyperbolic equations. Nevertheless, using tools of differential calculus, as well as computations on the Riemann problem (to be studied in the next parallel session), we get similar qualitative properties.

▸ Thermodynamic laws

First principle of thermodynamics :

The Hopf equation is nothing else than the conservation law

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0\,,$$

It will be the correct formulation for non smooth solutions.

Second principle of thermodynamics :

Smooth solutions further satisfy

$$\partial_t F(u) + \partial_x G(u) = 0\,,$$

for any functions F and G such that $G'(z) = zF'(z)$.
This identity is no more satisfied in the presence of singularities. A weakened version will give an admissibility criterion for singularities to be physically relevant.

3. The method of characteristics

- Transport equations and ODEs

The solutions to the **transport equation**

$$\partial_t v + a(t,x)\partial_x v = 0, \quad v_{|t=0} = v_0 \tag{2}$$

can be written simply in terms of the solutions to the **ordinary differential equation**

$$\frac{dX_{x_0}(t)}{dt} = a(t, X_{x_0}(t)), \quad X_{x_0}(0) = x_0\,. \tag{3}$$

We indeed have

$$v(t, X_{x_0}(t)) = v_0(x_0).$$

If $\mathbf{X}(t) : x_0 \mapsto X_{x_0}(t)$ is a bijection, then

$$v(t,x) = v_0((\mathbf{X}(t))^{-1}(x_0)).$$

In the case of a constant convection field a, the motion is uniform

$$v(t,x) = v_0(x - at)$$

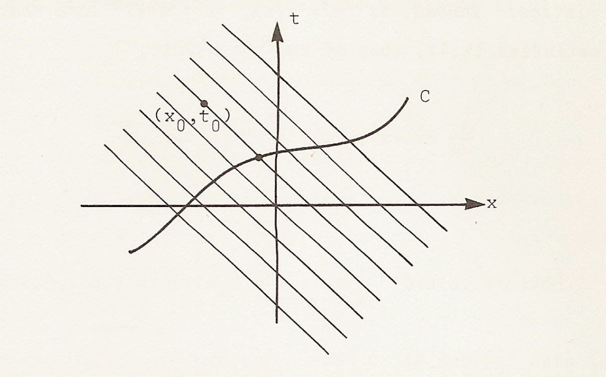

Fig. 5. Characteristics are straight lines

Under suitable regularity assumptions on a, the **Cauchy-Lipschitz theorem** ensures that the trajectories of (3) are locally well-defined and unique, so that $\mathbf{X}(t)$ is invertible.

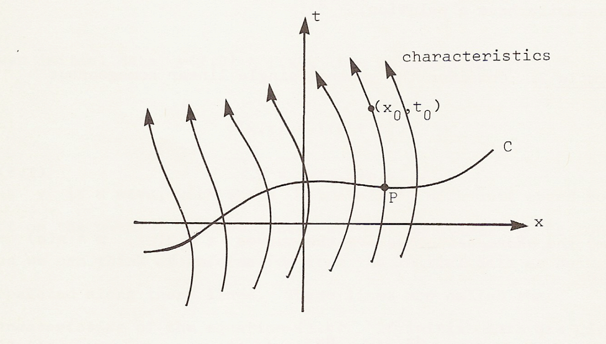

Fig. 6. Characteristics give a mapping from $\mathbb{R}$ to $\mathbb{R}$

▸ Characteristics associated to the Hopf equation

In the case of the Hopf equation, the velocity field u is transported by itself. We thus have

$$\begin{aligned} \frac{dX_{x_0}(t)}{dt} = u(t, X_{x_0}(t)), \quad X_{x_0}(0) = x_0, \\ u(t, X_{x_0}(t)) = u_0(x_0). \end{aligned} \tag{4}$$

as long as $\mathbf{X}(t) : x_0 \mapsto X_{x_0}(t)$ is a smooth change of variables.

Note that the fact that $\mathbf{X}(t)$ is a smooth change of variables is related to the regularity of u (by the Cauchy-Lipschitz theorem).

▸ Finite time breakdown

Differentiating the equation of characteristics with respect to x_0, we get

$$\frac{d}{dt}\frac{dX_{x_0}(t)}{dx_0} = \frac{d}{dx_0}(u(t, X_{x_0}(t))) = u_0'(x_0)$$

Integrating with respect to t, we obtain

$$\frac{dX_{x_0}(t)}{dx_0} = 1 + u_0'(x_0)t.$$

At time

$$t_* = \frac{1}{\max_{x\in\mathbb{R}}(-u_0'(x))_+} = \frac{1}{(-u_0'(x_*))_+}$$

$\mathbf{X}(t_*)$ is no more a smooth change of variables :
at point (t_*, x_*) characteristics intersect and thus $u(t_*, x_*)$ is not defined.

Furthermore, in general, there is a jump discontinuity

$$\lim_{x\to x_*^-} u(t_*, x) \neq \lim_{x\to x_*^+} u(t_*, x).$$