Solutions in the sense of distributions
Definition, non uniqueness

1. Notion of distributions

In order to build weak solutions to the Hopf equation, we need to define derivatives of non smooth functions, typically of Heaviside functions. The suitable tool to do that is Schwartz' theory of distributions.

▸ Duality

The space of distributions is essentially the smaller space

- containing continuous functions,
- and stable by derivation.

To give a precise sense to this definition, we have to change our point of view on functions. Instead of considering a function $f : \mathbb{R} \to \mathbb{R}$ as the collection of its values $f(x)$ for all $x \in \mathbb{R}$, we define f by its **averages against suitable test functions** $\int f(x)\varphi(x)dx$.

The class of **test functions** is $\mathcal{D} = C_c^\infty(\mathbb{R})$, so that, for any $\varphi \in \mathcal{D}$, one can define

- infinitely many derivatives ;
- the integral against any continuous function.

Distributions are "continuous " linear forms on $\mathcal{D}$.

Any **continuous function** u **can be identified as a distribution**

$$T_u : \varphi \in \mathcal{D} \mapsto \langle T_u, \varphi \rangle = \int \varphi(x) u(x) dx \,.$$

Results from integration theory guarantees that

$$u \in C^0(\mathbb{R}) \mapsto T_u \in \mathcal{D}$$

is an injective mapping.

▸ Derivation of distributions

Weak derivation has to correspond to classical derivation if relevant. For any function $u \in C^1(\mathbb{R})$, an integration by part shows that

$$\langle T_{\partial u}, \varphi \rangle = \int \varphi(x) u'(x) dx = - \int \varphi'(x) u(x) dx \,,$$

since there is no contribution of the boundary terms.

By definition, the **derivative** $\partial^\alpha T$ of order α of the distribution T is therefore the distribution defined by

$$\forall \varphi \in \mathcal{D}, \quad \langle \partial^\alpha T, \varphi \rangle \stackrel{\text{def}}{=} (-1)^\alpha \langle T, \partial^\alpha \varphi \rangle \,.$$

In particular, we define the “weak derivative” of a continuous function as the derivative of the corresponding distribution.

▸ Examples

- The **Heaviside function** is the function H defined on $\mathbb{R}$ by

$$H(x) = 0 \text{ if } x \leq 0, \quad H(x) = 1 \text{ if } x > 0.$$

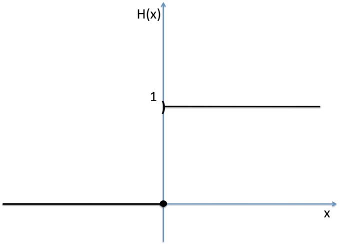

Fig. 1. The Heaviside function

It is of course C^∞ on $\mathbb{R}_*^+$ and on $\mathbb{R}_*^-$, but has a jump at $x = 0$, so that it is even not continuous....
It has nevertheless a weak derivative defined by

$$\langle \partial H, \varphi \rangle = -\int_{-\infty}^{+\infty} H(x)\varphi'(x)dx = \varphi(0)$$

- The **Dirac mass** is the distributional derivative of H

$$\delta = \partial_x H$$

It is NOT a function, but it can be approximated (as all distributions) by sequences of functions, for instance $\frac{1}{\varepsilon}\psi\left(\frac{x}{\varepsilon}\right)$ where ψ is some smooth function with compact support such that $\int \psi(x)dx = 1$.

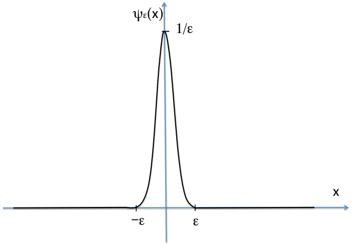

Fig. 2. Approximation of the Dirac mass

$$\int \frac{1}{\varepsilon}\psi\left(\frac{x}{\varepsilon}\right)\varphi(x)dx = \int \psi(y)\varphi(\varepsilon y)dy = \varphi(0) + \int \psi(y)\left(\varphi(\varepsilon y) - \varphi(0)\right)dy$$

tends indeed to $\varphi(0)$ as $\varepsilon \to 0$ (since φ' is uniformly bounded).

2. Distributional solutions of the Hopf equation

- The unknown

We can now extend the notion of solution of a PDE by considering derivatives in weak sense. This is actually a very standard tool for linear PDEs.

But **products of distribution** are not defined in general !
δ is not a function, therefore δ^2 does not make sense
H is not continuous at $x = 0$, therefore δH does not make sense

In order that the Hopf equation makes sense, we therefore require that **weak solutions are bounded functions**.

▶ The conservative form of the equation

For such functions, the **equation in conservative form**

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0$$

is well-defined.

Note that this is in general not the case of the **transport equation**

$$\partial_t u + u\partial_x u = 0.$$

Both formulations are not equivalent !

For the same reason, the **conservation of entropy**

$$\frac{1}{2}\partial_t u^2 + \frac{1}{3}\partial_x u^3 = 0\,,$$

is not an equivalent formulation, and will not be satisfied in general for weak solutions of the Hopf equation.

▸ The integral formulation

We call **solution in the sense of distributions** of the Hopf equation any function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ (defined and bounded almost everywhere) such that for all $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$

$$\iint \left(u \partial_t \varphi + \frac{1}{2} u^2 \partial_x \varphi \right)(t,x)\, dx dt = - \int u_0 \varphi_{|t=0} dx \,.$$

Since distributions are defined by averages, changing the values of a bounded function on a countable number of points does not modify the corresponding distribution. This is why weak solutions are defined only **almost everywhere**.

Note that test functions are compactly supported on $\mathbb{R}^+ \times \mathbb{R}$, so that boundary terms appear in the integration by parts. The **initial condition** is therefore encoded in the integral formulation.

▸ Some solutions

Strong solutions obtained by the method of characteristics are distributional solutions of the Hopf equation. For piecewise C^1 functions, one can indeed justify the integration by parts

$$\iint (u\partial_t\varphi + \frac{1}{2}u^2\partial_x\varphi)dtdx = -\iint (\partial_t u + \frac{1}{2}\partial_x u^2)\varphi dtdx - \int u\varphi(0,x)dx$$
$$= -\int u_0\varphi(0,x)dx$$

Rarefaction waves are distributional solutions of the Hopf equation

$$u(t,x) = v\left(\frac{x}{t}\right) \text{ with } v(z) = \max(u_l, \min(z, u_r))\,.$$

The only additional difficulty here is the singularity at time 0, which can be dealt with by approximation

$$\int_\varepsilon^{+\infty}\int (u\partial_t\varphi + \frac{1}{2}u^2\partial_x\varphi)dtdx = -\int u(\varepsilon,x)\varphi(\varepsilon,x)dx$$
$$\to -\int_{-\infty}^0 u_l\varphi(0,x)dx - \int_0^{+\infty} u_r\varphi(0,x)dx$$

Shock waves are distributional solutions of the Hopf equation.

$$u(t,x) = v\left(\frac{x}{t}\right) \text{ with } v(z) = u_l \mathbb{1}_{z \le s} + u_r \mathbb{1}_{z>s}\,.$$

A straightforward computation indeed leads to

$$\begin{aligned}
&\iint (u\partial_t\varphi + \frac{1}{2}u^2\partial_x\varphi)dtdx \\
&\qquad = \iint_{x\le st}(u_l\partial_t\varphi + \frac{1}{2}u_l^2\partial_x\varphi)dtdx + \iint_{x>st}(u_r\partial_t\varphi + \frac{1}{2}u_r^2\partial_x\varphi)dtdx \\
&\qquad = -\int_{x>0} u_l\varphi\left(\frac{x}{s},x\right)dx - \int_{x<0} u_l\varphi(0,x)dx + \int\frac{1}{2}\int u_l^2\varphi(t,st)dt \\
&\qquad + \int_{x>0} u_r\varphi\left(\frac{x}{s},x\right)dx - \int_{x>0} u_r\varphi(0,x)dx - \int\frac{1}{2}\int u_r^2\varphi(t,st)dt \\
&\qquad = \int\Big(-s(u_l-u_r)+\frac{1}{2}(u_l^2-u_r^2)\Big)\varphi(t,st)dt - \int u_0(x)\varphi(0,x)dx
\end{aligned}$$

The Hopf equation is then equivalent to the Rankine-Hugoniot jump conditions giving the shock speed :

$$-s(u_l-u_r)+\frac{1}{2}(u_l^2-u_r^2)=0\,.$$

3. About existence and uniqueness

▸ Global existence of solutions for bounded initial data

The existence of weak solutions to the Hopf equation does not result from a general theorem (like the Cauchy-Lipschitz theorem for ODEs, or some fixed point theorem). Weak solutions are built by approximation :

- **Glimm's approximation scheme** described in the previous session
  Discretization of the Hopf equation with respect to space and time, and explicit resolution of the Riemann problem.
- **Viscous approximation** to be studied in the next parallel session
  Smoothing of the Hopf equation by some viscous dissipation, and explicit resolution of the heat equation.
- **Kinetic formulation**
  Statistical description of the microscopic structure of the fluid, and study of the relaxation process

Proving the **consistence** of these approximations is not really difficult : the approximate solution satisfies the Hopf equation up to a small remainder.

The point is to get the **convergence**, i.e. the stability of the Hopf equation : the approximate solution is close (at least in some weak sense) to some true solution.

The difficulty comes from the **nonlinearity**. Weak convergence (which is more or less some convergence in average) is not enough to describe the asymptotic behaviour of nonlinear terms.

The main possible pathologies come from **oscillations and concentrations**, which have to be excluded by additional a priori estimates.

▸ Non uniqueness

With that notion of solution, we have **no more uniqueness** !
Starting from instance of the Heaviside function $u_0 = H$, we can check that both functions u_1 and u_2 defined respectively by

$$u_1(t,x) = H\left(x - \frac{1}{2}t\right),$$

$$u_2(t,x) = \begin{cases} 0 \text{ if } x < 0 \\ \frac{x}{t} \text{ if } 0 < x < t \\ 1 \text{ if } x > t \end{cases}$$

are solutions in the sense of distributions of the Hopf equation, i.e.

$$\iint \left(u\partial_t\varphi + \frac{1}{2}u^2\partial_x\varphi\right) dxdt = -\int u_0(x)\varphi(0,x)dx\,.$$

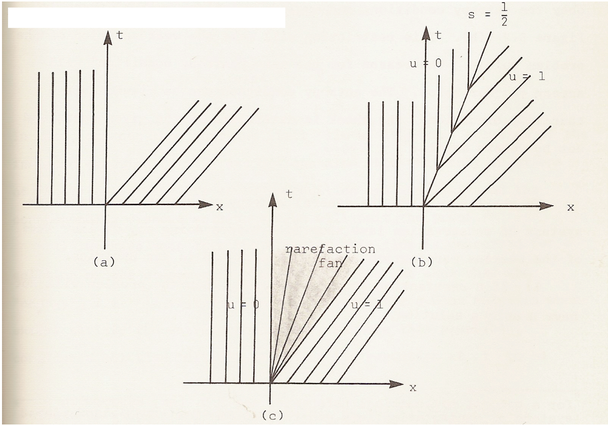

Fig. 3. Non uniqueness of distributional solutions

The Riemann problem with $u_l < u_r$ is underdetermined. On can find at least two weak solutions, a rarefaction wave and a shock wave.

▸ Physical admissibility conditions

The shock wave can be discarded using

- Lax-Oleinik's condition

$$t\partial_x u \leq 1 \text{ in the sense of distributions;}$$

- some entropy inequality, for instance

$$\frac{1}{2}\partial_t u^2 + \frac{1}{3}\partial_x u^3 \leq 0 \text{ in the sense of distributions;}$$

- a causality principle : characteristics coming from the shock should be traced back to the initial time.

These conditions are actually inherited from the **microscopic structure of shocks**. Anyone of them ensures uniqueness of the solution starting from any bounded initial data.